Topological Characteristics of Random Surfaces Generated by Cubic Interactions

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Abstract: We consider random topologies of surfaces generated by cubic interactions. Such surfaces arise in various contexts in 2-dimensional quantum gravity and as world-sheets of string theory. Our results are most conveniently expressed in terms of a parameter $h = n/2 + \chi$, where n is the number of interaction vertices and χ is the Euler characteristic of the surface. Simulations and results for similar models suggest that $\mathrm{Ex}[h] = \log(3n) + \gamma + O(1/n)$ and $\mathrm{Var}[h] = \log(3n) + \gamma - \pi^2/6 + O(1/n)$. We prove rigourously that $\mathrm{Ex}[h] = \log n + O(1)$ and $\mathrm{Var}[h] = O(\log n)$. We also derive results concerning a number of other characteristics of the topology of these random surfaces.

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1. Introduction

Many years ago, Wheeler [1] argued that in a theory of quantum gravity, large fluctuations in curvature at small distance scales, mandated by the uncertainty principle, would give rise to corresponding fluctuations in the topology of spacetime. Thus, though spacetime at large distance scales appears to have the simple topology of a ball in Euclidean space, at small distance scales its topology may be that of a complicated and dynamically changing space-time foam. Some of the consequences of space-time foam have been explored by Hawking [2,3] and others (see for example Carlip [4,5]) in the context of Euclidean quantum gravity. Although this work has shed light on the qualitative aspects of topological fluctuations, it is difficult to obtain quantitative results, such as the probability distributions of topological invarants, in four or even three dimensions in other than very special cases. Indeed, even recognizing topological equivalence (diffeomorphism) is undecidable for four-manifolds (see Markov [6], Boone *et al.* [7] and, for an overview of this issue and its relationship to quantum gravity, Schleich and Witt [8]), and an open problem for three-manifolds. On the other hand, two-dimensional topology is sufficiently tractible that quantitative results can easily be obtained for a variety of models. Much of this previous work has focused on models in which the contributions from the two dimensional topology enters into the theory weighted in a prescribed fashion by other fields. In contrast, this paper will explore a two-dimensional model giving rise to a probability distribution on the topology alone, independent of any geometric or dynamical considerations.

Hermitian matrix models provide a prominent example in which the contribution of two dimensional topology to quantum amplitudes appears in conjunction with that of other fields. The general form of the free energy for such a model is given by

$$\log Z = \log \int \exp -\mathrm{Tr}\big(V(\Phi)\big)\, d\Phi,$$

where the integral is over all $N \times N$ Hermitian matrices Φ, and $V(\Phi)$ is a polynomial in Φ. The large-N limit, in which $N \to \infty$ with the coefficients of V fixed, has been shown by 't Hooft [9, 10] to be dominated by the dynamics of the fields on a surface of genus zero (see also Brezin *et al.* [11] and Bessis *al.* [12]). The double-scaling limit, in which the coupling constant in V varies in a prescribed way as $N \to \infty$, includes contributions to the partition function from surfaces of all genera (see Brezin and Kazakov [13], Douglas and Shenker [14] and Gross and Migdal [15, 16]). One such model is equivalent to Euclidean quantum gravity in two dimensions, which corresponds to a probability distribution in which each closed compact connected two-dimensional manifold is weighted by the exponential of its Euler characteristic (see also the survey of Di Francesco *et al.* [17]).

String theory provides another example of a theory in which contribution of the topology of two-dimensional surfaces to quantum amplitudes enters coupled to their geometry, as induced by an embedding in a higher-dimensional space-time. In perturbative string theory, interacting strings have higher-genus worldsheets (see Polyakov [18] and Alvarez [19], as well as Polchinski [20] (pp. 86–90) and the references therein). In string field theory, Witten [21] proposed a cubic interaction for open strings for which these worldsheets are two-dimensional surfaces with boundary (see also Horowitz *et al.* [22]). Note that string theories have natural connections to matrix models; for example, topological strings provide a natural connection between supersymmetry in four dimensions and matrix models (see Ooguri and Vafa [23]).

There are also a number of approaches that treat topology in a discrete way. In particular, Regge [24] introduced Regge calculus, in which a discrete triangulation forms the framework underlying a piecewise flat approximation to a continuous geometry. This approach has proven to be a useful framework for the study of many issues in classical and quantum gravity in two and more dimensions (see the surveys of Williams and Tuckey [25] and Williams [26]). A special instance of this approach, dynamical triangulations, has also been extensively used to investigate properties of quantum gravity (see also Ambjørn *et al.* [27]). Interestingly enough, in two dimensions, Weingarten [28], Ambjørn *et al.* [29], Frohlich [30], David [31, 32] and Kazakov *et al.* [33] have shown that certain dynamical triangulation models have deep connections with string theory. All of these approaches have the property that the topology is prescribed *a priori*, so that only quantities related to geometry can be predicted by the theory. However, there are some generalizations of Regge calculus that lead to theories that can allow consideration of different topologies. Ponzano and Regge [34] introduced a particular discretization of geometry through spin variables (elements of $SU(2)$) associated with the edges of the triangulation. Their extension is specific to three dimensions, but work of Penrose [35, 36, 37] and others (see Barrett and Crane [38] and Baez [39, 40]) on spin networks has generalized it to higher dimensions by associating spin variables (in $SU(2)$ and other groups) with two-dimensional simplices. Certain formulations of spin networks allow consideration of the contribution of different topologies. However, as in string theory and matrix models, topological and spin variable contributions enter together in the computation of physical quantities.

In an interesting approach closely related to our model, Hartle [41] has formulated a two dimensional Regge calculus model in which one sums over topologies as well as geometries in computing amplitudes, and which treats the topology in a discrete fashion in a form separable from its geometry. Hartle then considers the contribution of pseudomanifolds

to quantum amplitudes by calculating the probability distribution of pseudomanifolds for small triangulations.

The model explored in this paper shares with string theory a cubic interaction, corresponding to graphs in which each vertex is incident with three edges. Our model is also related to a matrix model with $N = 1$ and V a cubic polynomial. However, unlike these examples, our model focusses on the discrete space of topologies of surfaces, ignoring geometric and other structure. As elucidated in Section 2, our model is entirely discrete, and can be formulated combinatorially in terms of graphs or permutations. Our model shares with Hartle's model this discrete nature. However, it differs from Hartle's model in three aspects: 1) Hartle's model assigns positive probabilities to pseudomanifolds whereas we consider only manifolds, 2) Hartle's model assigns equal probabilities to all pseudomanifolds with a given number of vertices whereas we assign equal probabilites to all manifolds with a given number of triangles, and 3) Hartle's model requires that the pseudomanifolds be simplicial whereas we do not require this of our triangulations. These differences allow us to examine the probability distribution of our model for large triangulations.

2. Probabilistic Models

We shall focus our attention on one particular probability distribution, but there are several mathematical models giving rise to this distribution. The simplest of these models, the *quotient model*, produces a random orientable surface as the quotient surface of a number of solid triangles. We shall refer to the elements of the boundaries of these triangles as *corners* and *arcs*, rather than as vertices and edges (since we shall reserve the terms vertices and edges for elements in the thin graph model presented below). Take the $3n$ arcs of an even number n of oriented triangles and identify them in pairs, respecting the orientation, so that the resulting surface is orientable, and with all $(3n-1)!! = (3n-1)\cdot(3n-3)\cdots 3\cdot 1$ pairings being equally likely. (These pairing may identify two arcs from the same triangle, or more than one pair of arcs between two triangles.) The surface is triangulated with n triangles, and their $3n$ arcs before identification become $3n/2$ arcs after identification. Let the random variable h denote the number of corners after identification, and let χ denote the Euler characteristic of the surface. Then we have $\chi = h - 3n/2 + n$, so that $h = \chi + n/2$. The distribution of h will be the main object of study in this paper, but studying h is equivalent to studying χ. We note that, since χ is even, h has the same parity as $n/2$. We shall see in Section 3 that with probability $1 - 5/18n + O(1/n^2)$ the surface is connected, and thus consists of a single component. In this case, the genus g

of the surface is given by $g = 1 - \chi/2 = 1 + n/4 - h/2$. Thus in this case studying h is equivalent to studying g.

A variant of the quotient model is the *fat-graph model*. Take n triangles as before, but instead of identifying their arcs in pairs, join them in pairs with rectangular *ribbons*, again in such a way that the resulting surface is orientable, and again with all $(3n-1)!!$ pairings being equally likely. The resulting surface in this case will have a non-empty boundary, comprising one or more *boundary cycles*. If we shrink each boundary cycle to a point (or if we cap off each boundary cycle with an appropriate polygon), we obtain a surface topologically equivalent to that produced by the quotient model for the same pairing. The points produced by shrinking boundary components correspond to the corners after identification in the quotient model, so that the number of boundary components is h. We shall exploit this correspondence in Sections 4 and 5 to study h by analyzing an algorithm that traces out the boundary cycles in the fat-graph model.

Another variant of the quotient model is the *thin-graph model*. In the quotient surface, install a *vertex* corresponding to each triangle and install an *edge* joining two vertices for each pair of arcs of the corresponding triangles that are identified. (Alternatively, one may shrink each triangle in the fat graph to a vertex, and shrink each ribbon to an edge. In any case, the graph may have "loops" (edges joining vertices to themselves) and "slings" (multiple edges joining a single pair of vertices).) The result is a cubic graph (each vertex has degree 3), with an additional structure: the three edges incident with each vertex have one of two possible cyclic orderings. As was observed by Heffter [42], these cyclic orderings allow the quotient surface or fat graph to be reconstructed from the thin graph. The thin graph may be regarded as a bubble diagram for a cubic interaction, and it is this viewpoint that forms the physical basis for our models. The thin graph, even without the cyclic ordering structure, reveals the number of connected components of the random surfaces in the quotient and fat-graph models. We shall exploit this fact in Section 3 to estimate the probability that these surfaces are connected.

A final variant of these models is the *permutation model*. Here we ignore surfaces and graphs and simply consider a probability distribution on permutations. Let ϱ be a permutation of $3n$ elements having cycle structure $[2^{3n/2}]$ (say $(12)\,(34)\cdots(3n{-}1\;3n)$), and let σ be a permutation having cycle structure $[3^n]$ (say $(123)\,(456)\cdots(3n-2\;3n-1\;3n)$). Let π be a random permutation, with all $(3n)!$ permutations being equally likely. Then $\pi\,\varrho\,\pi^{-1}$ is a random permutation uniformly distributed over the permutations with cycle structure $[2^{3n/2}]$. If we think of the cycles of σ as corresponding to the triangles of the fat graph model and those of $\pi\,\varrho\,\pi^{-1}$ as corresponding to the ribbons, we see that the

cycles of $\pi\,\varrho\,\pi^{-1}\,\sigma$ correspond to the boundary cycles in the fat graph. We shall exploit this correspondence in Section 6 to to study the probability that $h = 1$.

For $n = 2$ triangles, there are $5!! = 15$ possible pairings, but they fall into just three combinatorial equivalence classes. (Two pairings are combinatorially equivalent if they lie in the same orbit under the action of the symmetry group that permutes the triangles and cyclically permutes their boundary elements (without changing their orientation).) Three representative pairings for these equivalence classes are shown in Figure 1; alongside them are shown their corresponding thin graphs (or their fat graphs with very small triangles and very long and slender ribbons). The arrows indicate the cyclic orderings of the edges at each vertex. These three representatives are in classes of sizes 9, 3 and 3 (from top to bottom); the upper two yield spheres ($h = 3$), while the lowest one yields a torus ($h = 1$). Thus the probability of a sphere is $(9 + 3)/15 = 4/5$, while the probability of a torus is $3/15 = 1/5$.

There are two other mathematical models that give rise to different probability distributions, but which yield results similar to those we have observed for random surfaces generated by cubic interactions. The first of these is simply a random permutation of n elements, with all $n!$ permutations being equally likely. Let h' denote the number of cycles in such a random permutation. We note that h' may take any value from 1 to n. The probability distribution of h' is given by the generating function $\binom{\xi+n-1}{n}$, in which $\Pr[h' = k]$ is the coefficient of ξ^k. We have $\left(\frac{d}{d\xi}\binom{\xi+n-1}{n}\right)_{\xi=1} = H_n$, where $H_n = \sum_{1\le k\le n} \frac{1}{k} = \log n + \gamma + O(1/n)$ and $\gamma = 0.5772\ldots$ is Euler's constant, and $\left(\frac{d^2}{d\xi^2}\binom{\xi+n-1}{n}\right)_{\xi=1} = H_n^2 - H_n^{(2)}$, $H_n^{(2)} = \sum_{1\le k\le n} \frac{1}{k^2} = \pi^2/6 + O(1/n)$ and $\pi = 3.14159\ldots$ is the circular ratio. These results yield $\mathrm{Ex}[h'] = \log n + \gamma + O(1/n)$ and $\mathrm{Var}[h'] = \log n + \gamma - \pi^2/6 + O(1/n)$. We also note that $\Pr[h' = 1] = 1/n$.

Another analogous model was introduced by Harer and Zagier [43]. Instead of starting with n triangles, each having 3 arcs, they start with a single polygon having $2n$ arcs, and again identify the arcs in pairs to obtain an orientable surface, with all $(2n - 1)!!$ pairings being equally likely. Let h'' denote the number of equivalence classes of corners in the resulting quotient surface (or equivalently the number of boundary cycles in the resulting fat graph). We note that h'' has the opposite parity from n. They showed that the generating function for h'' is $\frac{1}{2}\sum_{l+m=n+1}\binom{\xi}{l}\binom{\xi+m-1}{m}$ by expressing this generating function as a ξ-fold integral over a Gaussian Hermitian ensemble. (Subseqently, Penner [44] derived this result using perturbative series, and Itzykson and Zuber [45] derived in two further ways, one using group representations and another expoiting an analogy with

the second quantization of the harmonic oscillator.) We have $\left(\frac{d}{d\xi}\binom{\xi}{l}\right)_{\xi=1} = 1$ if $l = 1$ and $(-1)^l/l(l-1)$ otherwise, and $\left(\frac{d^2}{d\xi^2}\binom{\xi}{l}\right)_{\xi=1} = 0$ if $l = 1$ and $2(-1)^{l-1}(H_{l-2} - 1)/l(l-1)$ otherwise. These results yield $\mathrm{Ex}[h''] = \log(2n) + \gamma + O(1/n)$ and $\mathrm{Var}[h''] = \log(2n) + \gamma - \pi^2/6 + O\big((\log n)/n\big)$. We also note that $\Pr[h'' = 1] = 1/(n+1)$.

We have conducted an empirical study of 10,000 random surfaces, each constructed from 80,000 triangles. The sample mean of h was 13.1092 and the sample variance was 11.3216 .... Since $\log(240{,}000) + \gamma = 12.9656\ldots$ and $\log(240{,}000) + \gamma - \pi^2/6 = 11.3206\ldots$, these results strongly suggest the conjecture that

$$\mathrm{Ex}[h] = \log(3n) + \gamma + O\left(\frac{1}{n}\right)$$

and

$$\mathrm{Var}[h] = \log(3n) + \gamma - \frac{\pi^2}{6} + O\left(\frac{1}{n}\right).$$

These conjectures may be compared with the results for the Harer-Zagier model: in each case the argument of the logarithm is the total number of arcs of the original polygons, and all remaining constant terms are identical. We have not been able to verify these conjectures at the stated level of precision, but in Section 4 we shall prove that

$$\mathrm{Ex}[h] = \log n + O(1)$$

and in Section 5 we shall prove that

$$\mathrm{Var}[h] = O(\log n).$$

Since $\mathrm{Var}[h]/\mathrm{Ex}[h]^2 \to 0$ as $n \to \infty$, this implies that the distribution of h is strongly concentrated about its mean.

In Section 6 we shall study the probability that $h = 1$. We use a technique based on representations of the symmetric group to show that show that

$$\Pr[h = 1] = \begin{cases} 0, & \text{if } n/2 \text{ even;} \\ \frac{2}{3n} + O\left(\frac{1}{n^2}\right), & \text{if } n/2 \text{ odd.} \end{cases}$$

This result may be compared with that for the Harer-Zagier model: in each case the probability, when it does not vanish, is asymptotic to the reciprocal of the number of edges in the thin graph.

In Section 7 we shall present a classification of the boundary cycles in the fat-graph model according to their "self-interactions", and give a heuristic estimation of the expected number of "simple" cycles of various orders. In Sections 8 and 9 we give exact expressions, and rigourously derive asymptotic estimates, for the number of simple cycles having the two lowest orders in this classification.

In the remainder of this paper, the variable n will always denote the number of triangles, and the random variable h will always denote the number of boundary cycles in the fat graph; all other variables may be used with different meanings from section to section. Unless otherwise indicated, the notation $O(\cdots)$ and $\Omega(\cdots)$ will refer to asymptotic behaviour as n tends to infinity through even integers.

3. The Probability of Connectedness

Theorem 3.1: Let c be the number of connected components of a random surface obtained from the quotient model with n triangles. Then

$$\Pr[c = 1] = 1 - \frac{5}{18n} + O\left(\frac{1}{n^2}\right).$$

Proof: We shall show that

$$\Pr[c \geq 2] = \frac{5}{18n} + O\left(\frac{1}{n^2}\right), \tag{3.1}$$

which is equivalent to the theorem. We shall work with the thin graph, which has the same number of connected components as the quotient surface.

First we show that

$$\Pr[c \geq 2] \leq \frac{5}{18n} + O\left(\frac{1}{n^2}\right). \tag{3.2}$$

We may assume $n \geq 4$, since since all cubic graphs on 2 vertices are connected. Let V be the set of n vertices of the thin graph. Say that a subset P of the vertices of the thin graph is *closed* of no edge of the thin graph joins a vertex in P to one in $V \setminus P$. (Thus a closed set is one that comprises one or more connected components of the thin graph.) The event $c \geq 2$ is equivalent to the existence of a closed set P with $\emptyset \neq P \neq V$. If P is such a closed set, then so is $V \setminus P$, and at least one of these sets must contain at most $n/2$ vertices. Thus

$$\Pr[c \geq 2] \leq \sum_{\#P \leq n/2} \Pr[P \text{ closed}].$$

A closed set must contain an even number of vertices, and there are $\binom{n}{2k}$ sets containing $2k$ vertices. The probability that a given set P containing $2k$ vertices is closed is

$$\frac{(6k-1)!!\,(3n-6k-1)!!}{(3n-1)!!},$$

since there are $(6k-1)!!$ ways of constructing a thin graph on P and $(3n-6k-1)!!$ ways of constructing a thin graph on $V \setminus P$. Thus we have

$$\Pr[c \geq 2] \leq \sum_{1 \leq k \leq n/4} F_k,$$

where

$$F_k = \frac{(6k-1)!!\,(3n-6k-1)!!}{(3n-1)!!}\binom{n}{2k}.$$

We shall show that F_k is a non-increasing function of k for $1 \leq k \leq n/2$. The ratio of consecutive terms,

$$R_k = \frac{F_{k+1}}{F_k} = \frac{(6k+5)(6k+1)(n-2k)}{(3n-2k-1)(3n-2k-5)(2k+2)},$$

is unity when the difference between its numerator and denominator,

$$S_k = (6k+5)(6k+1)(n-2k) - (3n-2k-1)(3n-2k-5)(2k+2),$$

vanishes. This cubic polynomial in k has roots at $k_0 = n/4 - 1/2$ and

$$k_{\pm} = \frac{3n - 6 \pm \sqrt{9n^2 + 36n + 16}}{12}.$$

Since $S_k \sim -144k^3$ for $k \to \pm\infty$, $S_k \leq 0$ and thus $R_k \leq 1$ for $k_- \leq k \leq k_0$. Since $n \geq 4$, we have $144n \geq 308$, which implies that $k_- \leq 1$. Thus we have $F_{k+1} \leq F_k$ for $1 \leq k$ and $k+1 \leq n/4$. This implies

$$\Pr[c \geq 2] \leq F_1 + F_2 + (n/2 - 2)F_3.$$

Since $F_1 = 5/18n + O(1/n^2)$, $F_2 = O(1/n^2)$ and $F_3 = O(1/n^3)$, this inequality yields (3.2).

Finally we show that

$$\Pr[c \geq 2] \geq \frac{5}{18n} + O\left(\frac{1}{n^2}\right). \tag{3.3}$$

To do this we focus our attention on closed sets containing 2 vertices. We have

$$\Pr[c \geq 2] \geq \sum_{P} \Pr[P \text{ closed}] - \sum_{P,Q} \Pr[P \text{ closed}, Q \text{ closed}],$$

where the first sum is over all P with $\#P = 2$, and the second sum is over all unordered pairs of distinct sets P and Q with $\#P = \#Q = 2$. The first sum contains $\binom{n}{2} = n^2/2 + O(n)$ terms, each equal to $5!!\,(3n-7)!!/(3n-1)!! = 5/9n^3 + O(1/n^4)$. For the second term, $\Pr[P \text{ closed}, Q \text{ closed}]$ vanishes unless P and Q are disjoint. Thus the second sum contains $\binom{n}{2}\binom{n-2}{2}/2 = O(n^4)$ non-vanishing terms, each equal to $5!!\,5!!\,(3n-13)!!/(3n-1)!! = O(1/n^6)$. Substituting these results in (4) yields (3).

Inequalities (3.2) and (3.3) yield (3.1), which completes the proof of the theorem. $\square$

4. The Expected Number of Cycles

Our main result is an estimate for the parameter h, regarded as the number of boundary cycles in the fat-graph model with n triangles.

Theorem 4.1: As n tends to infinity through even integers, we have

$$\mathrm{Ex}[h] = \log n + O(1).$$

Let c be one of the $3n$ corners of the original n triangles. Let p_k denote the probability that c lies in a boundary cycle of length k.

Proposition 4.2:

$$\mathrm{Ex}[h] = 3n \sum_{1 \leq k \leq 3n} \frac{p_k}{k}.$$

Proof: Let h_k denote the number of boundary cycles of length k. Then

$$\mathrm{Ex}[h] = \sum_{1 \leq k \leq 3n} \mathrm{Ex}[h_k].$$

Let c_k denote the number of corners in boundary cycles of length k. Then $c_k = k\,h_k$, so that

$$\mathrm{Ex}[h] = \sum_{1 \leq k \leq 3n} \frac{\mathrm{Ex}[c_k]}{k}. \tag{4.1}$$

The probability distribution is invariant under a group of symmetries that includes permutations of the triangles and cyclic permutations of the corners of each triangle. Since

this group acts transitively on the $3n$ corners, we have $\mathrm{Ex}[c_k] = 3n\,p_k$. Substituting this into (4.1) yields the proposition. $\square$

Our next step is to obtain estimates for p_k. This will be done through the following two lemmas.

Lemma 4.3: For $1 \le k \le n$, we have

$$p_k \le \frac{1}{3n-2k+1}\left(1+\frac{k}{3n-2k+5}\right).$$

Lemma 4.4: For $1 \le k \le n/2$, we have

$$p_k \ge \frac{1}{3n}\left(1-\frac{4k}{3n-2k+1}\right).$$

We shall prove these two lemmas below. For now, let us see how they combine to proove the following proposition.

Proposition 4.5: For $1 \le k \le n/2$, we have

$$\frac{3n\,p_k}{k} = \frac{1}{k} + O\left(\frac{1}{n}\right)$$

as n tends to infinity through even integers.

Proof: From Lemma 4.3, we have

$$\begin{aligned}
\frac{3n\,p_k}{k} &\le \frac{3n}{k}\cdot\frac{1}{3n-2k+1}\left(1+\frac{k}{3n-2k+5}\right) \\
&\le \left(\frac{1}{k}+\frac{2}{3n-2k+1}\right)\left(1+\frac{k}{3n-2k+5}\right) \\
&= \left(\frac{1}{k}+O\left(\frac{1}{n}\right)\right)\left(1+O\left(\frac{k}{n}\right)\right) \\
&= \frac{1}{k}+O\left(\frac{1}{n}\right).
\end{aligned}$$

From Lemma 4.4, we have

$$\begin{aligned}
\frac{3n\,p_k}{k} &\ge \frac{1}{k}\left(1-\frac{4k}{3n-2k+1}\right) \\
&= \frac{1}{k}+O\left(\frac{1}{n}\right).
\end{aligned}$$

Combining these bounds yields the proposition. $\square$

This proposition, together with Proposition 4.2, allows us to prove Theorem 4.1.

Proof of Theorem 4.1: Since the sum of the lengths of all boundary cycles is $3n$, there can be at most 5 boundary cycles of length exceeding $n/2$. Thus Proposition 4.1 yields

$$\mathrm{Ex}[h] = 3n \sum_{1 \le k \le n/2} \frac{p_k}{k} + O(1).$$

Evaluating the sum using Proposition 4.5 yields

$$\begin{aligned}\mathrm{Ex}[h] &= \sum_{1 \le k \le n/2} \left(\frac{1}{k} + O\left(\frac{1}{n} \right) \right) + O(1) \\ &= \log n + O(1),\end{aligned}$$

since $\sum_{1 \le k \le n/2} \frac{1}{k} = \log n + O(1)$. $\Box$

It remains to prove Lemmas 4.3 and 4.4. To do this, we shall analyze the following randomized procedure *cycle*, which constructs the boundary cycle containing the corner c, and returns its length as *cycle*(c).

Since we are dealing with orientable surfaces, we may regard them as two-sided, and may imagine one side coloured green and the other coloured orange. We shall regard the arcs of the triangles as being directed clockwise as seen from the green side. For any corner d of a triangle T, let d^- denote the preceding corner and d^+ the following corner in the boundary of T, so that (d, d^+), (d^+, d^-) and (d^-, d) are the directed arcs in the boundary of T.

Each ribbon installed by the following procedure is a quadrilateral containing four corners and four arcs. Two of these arcs come from the triangles joined by the ribbon; the other two will be called *links*. Links will be directed so that the arcs of a ribbon are directed counterclockwise as seen from the green side. Links are initially *unmarked*, but may subsequently be *marked* to indicate that they are part of the boundary cycle containing c.

The procedure uses a data structure called the *urn*. The urn contains at any time a set of corners. This set is initialized by putting the $3n$ corners into it. A specific corner may be *removed* from the urn, or a random corner may be *drawn* from the urn; the corner removed in this way is equally likely to be any of the corners currently in the urn.

integer procedure *cycle*(**corner** c);
begin
  corner *head*, *tail*, *next*, *point*;

```
    integer length;
    head := c;
    tail := c;
    length := 0;
    put the 3n corners into an urn;
    repeat
      remove head from the urn;
      next := draw from the urn;
      install a ribbon with arcs (head, next⁻), (next⁻, next), (next, head⁻)
        and (head⁻, head) in counterclockwise order as seen from the green side,
        and introduce (head, next⁻) and (next, head⁻) as unmarked links;
      while there is an unmarked link (head, point) do
        begin
          mark the link (head, point);
          length := length + 1;
          head := point
        end;
      while there is an unmarked link (point, tail) do
        begin
          mark the link (point, tail);
          length := length + 1;
          tail := point
        end;
    until head = tail;
    return length
  end
```

The **repeat** ... **until** statement is analogous to a **while** ... **do** statement, except that the body of the statement is executed before, rather than after, the condition is tested.

Figure 2 shows the three possibilites for the installation of the first ribbon; unmarked links are shown dashed, and marked links are shown bold. The most common case, in which the corner d drawn from the urn lies on a different triangle from c, is shown on the left; the special cases in which $d = c^-$ and $d = c^+$ are shown on the upper right and lower right, respectively.

The probability p_k that the corner c is in a boundary cycle of length k is simply the probability that the procedure invocation *cycle*(c) returns the value k. Our estimates for p_k will therefore be based on an analysis of this procedure.

Let us consider the subgraph formed by the corners and the links at some point in the execution of the procedure. By the *in-degree* of a corner we shall mean the number

(zero or one) of links directed into it, and by the *out-degree*, the number (also zero or one) directed out of it. This subgraph then comprises one or more components, each of which is either a *path* (a part of a boundary cycle) or a complete boundary cycle. Initially, there are $3n$ paths, each of length zero. As links are introduced, paths may be extended, merged or closed into cycles. In a path, the corner with out-degree zero is called the *front* and the corner with in-degree zero is called the *rear*. At any time, there is one marked path. Its front is called the *head* and its rear is called the *tail*. The variable *length* keeps track of the length of the marked path. Initially the head and tail coincide at c. When they again coincide after one or more ribbons have been installed, the marked path closes into a cycle, the procedure terminates by returning the length of this cycle. We observe that the length of the marked path increases by at least one at each execution fo the **repeat** ... **until** statement, but may increase by more if previously unmarked paths are merged at its head, or tail, or both.

To analyze the behaviour of the procedure, it will be convenient to represent its possible executions as a tree. This tree will comprise a number of *nodes*, each representing a state of execution of the procedure, joined by *branches*, each representing an execution of the body of the **repeat** ... **until** statement. One node, called the *root* of the tree, corresponds to the initial state of the procedure just before the first execution of the body of the **repeat** ... **until** statement. Other nodes, called the *leaves* of the tree correspond to the final states of the procedure after the last executions of the body of the **repeat** ... **until** statement. Each node other than the root has one or more *children*, corresponding to the states reach after vaious corners are drawn from the urn. Each node that is not a leaf has a unique *parent*, corresponding to the immediately preceding state in the execution. The *ancestors* and *descendents* of a node are defined in the obvious way.

With each node K we may associate a value $depth(K)$ (corresponding to the number of installed ribbons), as well as values $head(K)$, $tail(K)$ and $length(K)$, in the obvious way. Since each installation of a ribbon adds either one or two marked links, we have

$$depth(K) \leq length(K) \leq 2\,depth(K).$$

All nodes at the same depth d correspond to states in which the same number $3n - 2d$ of corners (including the current *head*) remain in the urn, and thus all of these nodes have the same number $3n - 2d - 1$ of children. If control reaches some node of the tree, it is equally likely to proceed to each of its children. We shall say that a node K is *shallow* if $depth(K) \leq k - 1$. If K is shallow, the probability of proceeding to any particular child of K is at least $1/3n$ and at most $1/(3n - 2k + 1)$. Finally, we shall call an internal node

K (that is, a node other than a leaf) *double* if $\mathit{head}(K)^- = \mathit{tail}(K)$, and call it *single* otherwise. If a node is double, then in proceeding to one of its children both newly added links will be marked, otherwise only one will be marked. (In either case, previously added links may also be marked.)

We are now erady to prove Lemmas 4.3 and 4.4.

Proof of Lemma 4.3: Let E_k denote the event that execution terminates at a leaf L with $\mathit{length}(L) = k$, so that $p_k = \Pr[E_k]$. If K is a node other than the root, let K^* denote its parent. We shall write

$$p_k = \Pr[E_k] = \Pr[E_k^1] + \Pr[E_k^2], \tag{4.2}$$

where E_k^1 (respectively, E_k^2) denotes the event that execution terminates at a leaf L such that $\mathit{length}(L) = k$ and L^* is single (respectively, double).

Let us first consider an upper bound for $\Pr[E_k^1]$. If E_k^1 occurs at L, we shall call L^* a *precursor* for E_k^1. If the node K is a precursor for E_k^1, then (1) K has exactly one child at which E_k^1 occurs (this child corresponds to drawing the corner $\mathit{tail}(K)^+$ from the urn as *next*), (2) $\mathit{length}(K) = k - 1$ (since proceeding to the child at which E_k^1 occurs will add one to *length*, resulting in $\mathit{length} = k$), and (3) K is shallow (since installing a ribbon increases *length* by at least one, so that $\mathit{depth}(K) \le \mathit{length}(K) = k - 1$). Letting K also denote the event that control reaches the node K, we have

$$\Pr[E_k^1 \mid K] = \frac{1}{3n - 2\mathit{depth}(K) - 1} \le \frac{1}{3n - 2k + 1}.$$

Along any path from the root to a leaf in the tree, at most one node can be a precursor to E_k^1 (since *length* strictly increases along any such path). Thus

$$\sum_{\substack{\text{precursor } K \\ \text{to } E_k^1}} \Pr[K] \le 1.$$

The last two bounds together yield

$$\begin{aligned}
\Pr[E_k^1] &= \sum_{\substack{\text{precursor } K \\ \text{to } E_k^1}} \Pr[E_k^1 \mid K] \Pr[K] \\
&\le \frac{1}{3n - 2k + 1} \sum_{\substack{\text{precursor } K \\ \text{to } E_k^1}} \Pr[K] \\
&\le \frac{1}{3n - 2k + 1}.
\end{aligned} \tag{4.3}$$

Next let us consider an upper bound for $\Pr[E_k^2]$. At a given node K, we shall say that a corner α *leads to* a corner β *in* l *steps* if α is the rear and β is the front of an unmarked path of length l at K. A corner γ such that γ^- leads to γ in l steps will be called a *reflector* of size l. If E_k^2 occurs at a leaf L, so that L^* is double, we shall call L^* a *precursor* for E_k^2. If K is a precursor for E_k^2, then (1) there exists an $l \geq 1$ such that all children of K at which E_k^2 occurs correspond to drawing reflectors of size l from the urn, and (2) $\mathit{length}(K) = k-l-2$ (since proceeding to a child at which E_k^2 occurs will add $l+2$ to *length*, resulting in $\mathit{length} = k$), and (3) $\mathit{depth}(K) \leq k-3$ (since $\mathit{depth}(K) \leq \mathit{length}(K) \leq k-3$).

If α is a reflector of size l at node K, and if $\mathit{length}(K) = k-l-2$, then K will be called a *precursor* for α. Let S_K^α be the event that corner α is drawn at node K. Then

$$\Pr[S_K^\alpha \mid K] = \frac{1}{3n - 2\mathit{depth}(K) - 1} \leq \frac{1}{3n-2k+5}. \tag{4.4}$$

Let R_J^α denote the event that corner α becomes a reflector at some child of node J. For any node J, there is at most one corner α (namely $\mathit{head}(J)^+$) that can become a reflector at a child of J, and if there is such an α, then there is just one child of J (namely the one corresponding to drawing the corner β from the urn, where β is the rear of the unmarked path with α as its front) at which α becomes a reflector. Thus, for any node J of depth at most $k-1$,

$$\sum_\alpha \Pr[R_J^\alpha \mid J] \leq \frac{1}{3n - 2\mathit{depth}(J) - 1} \leq \frac{1}{3n-2k+1}. \tag{4.5}$$

For E_k^2 to occur, some corner α must become a reflector of size l at a child of some shallow node J, and then at some descendant K of J that is a precursor for α, α must be drawn from the urn. Thus

$$\Pr[E_k^2] = \sum_{J \text{ shallow}} \Pr[J] \sum_\alpha \Pr[R_J^\alpha \mid J] \sum_{\substack{K \text{ precursor for } \alpha \\ K \text{ descendant of } J}} \Pr[K \mid J]\, \Pr[S_K^\alpha \mid K].$$

Using (4.4), we have

$$\Pr[E_k^2] \leq \frac{1}{3n-2k+5} \sum_{J \text{ shallow}} \Pr[J] \sum_\alpha \Pr[R_J^\alpha \mid J] \sum_{\substack{K \text{ precursor for } \alpha \\ K \text{ descendant of } J}} \Pr[K \mid J].$$

If α is a reflector of size l, a node K can be a precursor for α only if $length(K) = k - l - 2$. Since *length* strictly increases along any path descending from J, at most one node on any such path can be a precursor for α. Thus we have

$$\sum_{\substack{K \text{ precursor for } \alpha \\ K \text{ descendant of } J}} \Pr[K \mid J] \leq 1.$$

This yields

$$\Pr[E_k^2] \leq \frac{1}{3n - 2k + 5} \sum_{J \text{ shallow}} \Pr[J] \sum_{\alpha} \Pr[R_J^\alpha \mid J].$$

Using (4.5), we have

$$\Pr[E_k^2] \leq \frac{1}{3n - 2k + 1} \frac{1}{3n - 2k + 5} \sum_{J \text{ shallow}} \Pr[J].$$

Finally, since there are at most k nodes of depth at most $k - 1$ on any path from the root to a leaf in the tree, we have

$$\sum_{J \text{ shallow}} \Pr[J] \leq k. \tag{4.6}$$

This yields

$$\Pr[E_k^2] \leq \frac{1}{3n - 2k + 1} \frac{k}{3n - 2k + 5}.$$

Combining this inequality with (4.3) in (4.2) completes the proof of Lemma 4.3. □

Proof of Lemma 4.4: From (4.2), we have $\Pr[E_k] \geq \Pr[E_k^1]$, so it will suffice to obtain a lower bound to $\Pr[E_k^1]$. Let us say that a node K is a *strong precursor* to E_k^1 if (1) every ancestor of K (including K itself) is single, and (2) there are no reflectors at any ancestor of K, and (3) $length(K) = k - 1$, and (4) K is not a leaf. At a strong precursor K to E_k^1, there is exactly one child at which E_k^1 occurs. Thus, for a strong precursor K to E_k^1,

$$\Pr[E_k^1 \mid K] = \frac{1}{3n - 2\,depth(K) - 1} \geq \frac{1}{3n}.$$

Let A_k be the event that control reaches a strong precursor to E_k^1. Since *length* strictly increases along any path from the root to a leaf in the tree, at most one node on any such path can be a strong precursor to E_k^1. Thus

$$\Pr[A_k] = \sum_{\substack{K \text{ strong} \\ \text{precursor to } E_k^1}} \Pr[K].$$

It follows that

$$\begin{aligned}
\Pr[E_k] &\geq \Pr[E_k^1] \\
&\geq \sum_{\substack{K \text{ strong} \\ \text{precursor to } E_k^1}} \Pr[K] \Pr[E_k^1 \mid K] \\
&\geq \frac{1}{3n} \sum_{\substack{K \text{ strong} \\ \text{precursor to } E_k^1}} \Pr[K] \\
&= \frac{1}{3n} \Pr[A_k]. \qquad (4.7)
\end{aligned}$$

It remains to obtain a lower bound for $\Pr[A_k]$.

Let B_k denote the event that control reaches a shallow double node K. Let C_k denote the event that some corner becomes a reflector at a shallow node K. Let F_k denote the event that the procedure terminates at a leaf L with $\mathit{length}(L)$ less than k. Let G_k denote the event that control reaches a node J with $\mathit{length}(J)$ at least k, without ever passing through a node K with $\mathit{length}(K) = k - 1$. Then we have

$$\Pr[A_k] \geq 1 - \Pr[B_k] - \Pr[C_k] - \Pr[F_k, \overline{B_k}\,] - \Pr[G_k, \overline{B_k}, \overline{C_k}\,]. \qquad (4.8)$$

It remains to obtain upper bounds for the four probabilities on the right-hand-side.

First we deal with $\Pr[B_k]$. If a node K is double, but K^* is single, we shall call K^* a *double precursor*. If a node J is a double precursor, then (1) J is single, so that $\mathit{head}(J)^- \neq \mathit{tail}(J)$, and (2) J has exactly one child of J that is double (namely the one corresponding to drawing the corner $\mathit{head}(J)^-$ from the urn), and (3) $\mathit{head}(J)^+$ leads to $\mathit{tail}(J)^+$. Let M_J^α denote the event that at node J, control passes to a double child of J by drawing corner α from the urn. If B_k occurs, then M_J^α must occur for some α at some shallow node J. Thus

$$\Pr[B_k] \leq \sum_{J \text{ shallow}} \Pr[J] \sum_\alpha \Pr[M_J^\alpha \mid J].$$

Each term in the sum over α vanishes except possibly for the one with $\alpha = \mathit{head}(J)^-$, so that we have

$$\sum_\alpha \Pr[M_J^\alpha \mid J] \leq \frac{1}{3n - 2\,\mathit{depth}(J) - 1} \leq \frac{1}{3n - 2k + 1}.$$

This yields
$$\Pr[B_k] \le \frac{1}{3n-2k+1} \sum_{J \text{ shallow}} \Pr[J].$$
Using (4.6), we obtain
$$\Pr[B_k] \le \frac{k}{3n-2k+1}, \tag{4.9}$$
which is the desired upper bound for $\Pr[B_k]$.

Next we turn to $\Pr[C_k]$. We have
$$\Pr[C_k] \le \sum_{J \text{ shallow}} \Pr[J] \sum_{\alpha} \Pr[R_J^\alpha \mid J].$$
Using (4.5), we have
$$\Pr[C_k] \le \frac{1}{3n-2k+1} \sum_{J \text{ shallow}} \Pr[J],$$
and using (4.6), we obtain
$$\Pr[C_k] \le \frac{k}{3n-2k+1}, \tag{4.10}$$
which is the desired upper bound for $\Pr[C_k]$.

Next we deal with $\Pr[F_k, \overline{B_k}\,]$. If F_k occurs but B_k does not, then E_j^1 must occur for some $1 \le j \le k-1$. Thus
$$\Pr[F_k, \overline{B_k}\,] \le \sum_{1 \le j \le k-1} \Pr[E_j^1].$$
Using (4.3), we obtain
$$\Pr[F_k, \overline{B_k}\,] \le \frac{k}{3n-2k+1}, \tag{4.11}$$
which is the desired upper bound for $\Pr[F_k, \overline{B_k}\,]$.

Finally, we turn to $\Pr[G_k, \overline{B_k}, \overline{C_k}\,]$. At a given node J, a corner α will be called a *deflector* of size l if α^- leads to some corner β in l steps. Initially all deflectors are of size zero. As links are added, deflectors may be extended, merged into marked or unmarked paths or destroyed by being closed into unmarked cycles. A deflector α of size l will be called an *exit* from J if (1) $\mathit{length}(J) \le k-2$, and (2) $\mathit{length}(J) + l + 1 \ge k$. If G_k occurs but B_k and C_k do not, then at some shallow node J the corner drawn from the urn must be an exit from J. Thus
$$\Pr[G_k, \overline{B_k}, \overline{C_k}\,] \le \sum_{J \text{ shallow}} \Pr[J] \sum_{\substack{\text{exit } \alpha \\ \text{from } J}} \Pr[J^\alpha \mid J],$$

where J^α denotes the child of J reached by drawing α from the urn at J. Since J is shallow, we have

$$\Pr[j^\alpha \mid J] \le \frac{1}{3n - 2k + 1}.$$

This yields

$$\Pr[G_k, \overline{B_k}, \overline{C_k}\,] \le \frac{1}{3n - 2k + 1} \sum_{J \text{ shallow}} \Pr[J]\, X_J,$$

where X_J denotes the number of exits from J. If π is a path from the root to a leaf in the tree, we shall let

$$Y_\pi = \sum_{\text{shallow } J \text{ on } \pi} X_J$$

denote the number of exits from shallow nodes on π. Then we have

$$\sum_{J \text{ shallow}} \Pr[J]\, X_J = \sum_\pi \Pr[\pi]\, Y_\pi,$$

where the sum on the right-hand side is over all paths π from the root to a leaf in the tree, and $\Pr[\pi]$ is the probability that control follows the path π. Thus

$$\Pr[G_k, \overline{B_k}, \overline{C_k}\,] \le \frac{1}{3n - 2k + 1} \sum_\pi \Pr[\pi]\, Y_\pi. \tag{4.12}$$

We shall show that

$$Y_\pi \le k \tag{4.13}$$

for every path π. To do this, we shall charge each exit from a shallow node J on π against an unmarked link that is added at a shallow node on π, in such a way that at most one exit is charged against any link. Since there are at most $k-1$ shallow nodes on π, each of which adds at most one unmarked link, this will prove (4.13).

Suppose that corner α is an exit from node J on π. Let $s(\alpha, J)$ denote the size of the deflector α at J. Since α is an exit from J, we must have

$$\mathit{length}(J) + s(\alpha, J) + 1 \ge k,$$

so that the path from α^- contains at least $k - 1 - \mathit{length}(J) \ge 1$ links. We shall charge the exit α from J against the $\big(k - 1 - \mathit{length}(J)\big)$-th link (counting from the rear) in the path from α^-. It remains to verify that at most one exit from π is charged against any link. Suppose that exit α from node J is the first exit from π charged against link Λ. Any exit β from a later node K on π that is charged against a link on the path containing α^- will

be charged against a link that appears behind Λ on this path (since $\mathit{length}(K) \geq \mathit{length}(J)$ and any links in the path from α^- will also appear in the path from β^-). This completes the proof of (4.13).

From (4.12) and (4.13) we have

$$\Pr[G_k, \overline{B_k}, \overline{C_k}\,] \leq \frac{k}{3n - 2k + 1} \sum_{\pi} \Pr[\pi].$$

Since

$$\sum_{\pi} \Pr[\pi] \leq 1,$$

we obtain

$$\Pr[G_k, \overline{B_k}, \overline{C_k}\,] \leq \frac{k}{3n - 2k + 1}, \tag{4.14}$$

which is the desired upper bound for $\Pr[G_k, \overline{B_k}, \overline{C_k}\,]$.

Substituting (4.9), (4.10), (4.11) and (4.14) into (4.8), yields

$$\Pr[A_k] \geq 1 - \frac{4k}{3n - 2k + 1},$$

and substituting this into (4.7) completes the proof of Lemma 4.4. □

5. The Variance of the Number of Cycles

Theorem 5.1:

$$\mathrm{Var}[h] = O(\log n).$$

Proof: The proof is similar to that in Section 4, so we shall merely sketch it. We have

$$\mathrm{Var}[h] = \mathrm{Ex}[h^2] - \mathrm{Ex}[h]^2,$$

and we have seen that

$$\mathrm{Ex}[h] = \log n + O(1).$$

Thus to prove the theorem it will suffice to show that

$$\mathrm{Ex}[h^2] = (\log n)^2 + O(\log n). \tag{5.1}$$

We have

$$\mathrm{Ex}[h^2] = \sum_{1 \leq k \leq 3n} \sum_{1 \leq k' \leq 3n} \mathrm{Ex}[h_k \cdot h_{k'}],$$

where h_k denotes the number of cycles of length k.

Our first step is to reduce the range of the summations over k and k'. Since there are at most 5 cycles of length exceeding $n/2$, we have

$$\begin{aligned}\sum_{1\le k\le 3n}\sum_{1\le k'\le 3n}\mathrm{Ex}[h_k\cdot h_{k'}] &\le \sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\mathrm{Ex}[h_k\cdot h_{k'}]+10\mathrm{Ex}[h]+25\\ &= \sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\mathrm{Ex}[h_k\cdot h_{k'}]+O(\log n).\end{aligned}$$

Thus to prove (5.1) it will suffice to show that

$$\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\mathrm{Ex}[h_k\cdot h_{k'}]=(\log n)^2+O(\log n). \tag{5.2}$$

We have

$$\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\mathrm{Ex}[h_k\cdot h_{k'}]=\sum_{c}\sum_{c'}\sum_{1\le k\le n/2};\sum_{1\le k'\le n/2}\frac{\Pr[c\text{ in }k,c'\text{ in }k']}{k\cdot k'},$$

where the outer sums are over all corners and the event “corner c is in a cycle of length k” has been abbreviated “c in k”.

If c and c' are corners of the same triangle, we shall write $c\sim c'$, otherwise $c\not\sim c'$. We have

$$\begin{aligned}&\sum_{c}\sum_{c'\sim c}\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\frac{\Pr[c\text{ in }k,c'\text{ in }k']}{k\cdot k'}\\ &\qquad=\sum_{c}\sum_{c'\sim c}\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\frac{\Pr[c\text{ in }k]\ \Pr[c'\text{ in }k'\mid c\text{ in }k]}{k\cdot k'}\\ &\qquad\le\sum_{c}\sum_{c'\sim c}\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\frac{\Pr[c\text{ in }k]\ \Pr[c'\text{ in }k'\mid c\text{ in }k]}{k}\\ &\qquad\le\sum_{c}\sum_{c'\sim c}\sum_{1\le k\le n/2}\frac{\Pr[c\text{ in }k]}{k}\\ &\qquad\le 3\sum_{c}\sum_{1\le k\le n/2}\frac{\Pr[c\text{ in }k]}{k}\\ &\qquad\le 3\,\mathrm{Ex}[h]\\ &\qquad=O(\log n),\end{aligned}$$

since $\sum_{1\le k'\le n/2}\Pr[c' \text{ in } k' \mid c \text{ in } k] \le 1$, and for any corner c, there are just 3 corners c' such that $c \sim c'$. Thus to prove (5.2) it will suffice to show that

$$\sum_{c}\sum_{c'\not\sim c}\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in } k']}{k\cdot k'} = (\log n)^2 + O(\log n). \tag{5.3}$$

The probability distribution is invariant under a group of symmetries that includes permutations of the triangles and cyclic permutations of the corners of each triangle. Since this group acts transitively on the $9n(n-1)$ pairs (c, c') of corners such that $c \not\sim c'$, we have

$$\sum_{c}\sum_{c'\not\sim c}\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in } k']}{k\cdot k'}$$
$$= 9n(n-1)\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in } k']}{k\cdot k'},$$

where (c, c') is an arbitrary pair of corners such that $c \not\sim c'$. Since k and k' now appear symmetrically in the last sum, we have

$$9n(n-1)\sum_{1\le k\le n/2}\sum_{1\le k'\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in } k']}{k\cdot k'}$$
$$\le 18n(n-1)\sum_{1\le k\le n/2}\sum_{k\le k'\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in } k']}{k\cdot k'}.$$

Thus to prove (5.3) it will suffice to show that

$$18n(n-1)\sum_{1\le k\le n/2}\sum_{k\le k'\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in } k']}{k\cdot k'} = (\log n)^2 + O(\log n). \tag{5.4}$$

Now we shall express the event "corner c is in a cycle of length k and corner c' is in a cycle of length k'" as the disjoint union to two events: "corner c is in a cycle of length k and corner c' is in a disjoint cycle of length k'", which will be abbreviated "c in k, c' in disjoint k'", and "corner c is in a cycle of length k and corner c' is in the same cycle of length k'", which can occur only if $k = k'$ and which will be abbreviated "c in k, c' in same k". We may also refer to the events "c' in disjoint k'" and "c' in same k" conditioned on the event "c in k". We have

$$18n(n-1)\sum_{1\le k\le n/2}\sum_{k\le k'\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in } k']}{k\cdot k'}$$
$$= 18n(n-1)\sum_{1\le k\le n/2}\sum_{k\le k'\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in disjoint } k']}{k\cdot k'}$$
$$+ 18n(n-1)\sum_{1\le k\le n/2}\frac{\Pr[c \text{ in } k, c' \text{ in same } k]}{k^2}.$$

For the last sum we have

$$
\begin{aligned}
18n(n-1) &\sum_{1 \le k \le n/2} \frac{\Pr[c \text{ in } k, c' \text{ in same } k]}{k^2} \\
&= 18n(n-1) \sum_{1 \le k \le n/2} \frac{\Pr[c \text{ in } k] \Pr[c' \text{ in same } k \mid c \text{ in } k]}{k^2} \\
&\le 6n \sum_{1 \le k \le n/2} \frac{\Pr[c \text{ in } k]}{k} \\
&\le 2\mathrm{Ex}[h] \\
&= O(\log n),
\end{aligned}
$$

since $\Pr[c' \text{ in same } k \mid c \text{ in } k] \le k/3(n-1)$ (there are at most $k-1$ corners of the cycle containing c on different triangles from c, and the probability that a particular corner c' of the $3(n-1)$ corners on triangles different from c is among them is thus at most $(k-1)/3(n-1) \le k/3(n-1)$). Thus to prove (5.4) it will suffice to show that

$$
18n(n-1) \sum_{1 \le k \le n/2} \sum_{k \le k' \le n/2} \frac{\Pr[c \text{ in } k, c' \text{ in disjoint } k']}{k \cdot k'} = (\log n)^2 + O(\log n). \tag{5.5}
$$

We have

$$
\begin{aligned}
18n(n-1) &\sum_{1 \le k \le n/2} \sum_{k \le k' \le n/2} \frac{\Pr[c \text{ in } k, c' \text{ in disjoint } k']}{k \cdot k'} \\
&= 6n \sum_{1 \le k \le n/2} \frac{\Pr[c \text{ in } k]}{k} \, 3(n-1) \sum_{k \le k' \le n/2} \frac{\Pr[c' \text{ in disjoint } k' \mid c \text{ in } k]}{k'}.
\end{aligned} \tag{5.6}
$$

We can estimate $\Pr[c' \text{ in disjoint } k' \mid c \text{ in } k]$ the same way as we estimated $\Pr[c \text{ in } k]$ in Lemma 4.3 of the deriviation of $\mathrm{Ex}[h]$. The only difference is that we start with a situation in which up to k ribbons may already have been installed. Thus we must replace k by $k + k' \le 2k'$ in Lemma 4.3:

$$
\Pr[c' \text{ in disjoint } k' \mid c \text{ in } k] \le \frac{1}{3n - 4k' + 1} \left(1 + \frac{2k'}{3n - 4k' + 5} \right).
$$

Thus for the inner sum in (5.6) we have

$$
\begin{aligned}
3(n-1) &\sum_{k\le k'\le n/2} \frac{\Pr[c' \text{ in disjoint } k' \mid c \text{ in } k]}{k'} \\
&\le 3(n-1) \sum_{k\le k'\le n/2} \frac{1}{k'}\cdot\frac{1}{3n-4k'+1}\left(1+\frac{2k'}{3n-4k'+5}\right) \\
&\le \sum_{k\le k'\le n/2} \left(\frac{1}{k'}+\frac{4}{3n-4k'+1}\right)\left(1+\frac{2k'}{3n-4k'+5}\right) \\
&= \log n - \log k + O(1).
\end{aligned}
$$

Furthermore, we still have

$$
\Pr[c \text{ in } k] \le \frac{1}{3n-2k+1}\left(1+\frac{k}{3n-2k+5}\right)
$$

from Lemma 4.3. Thus for the outer sum in (5.6) we have

$$
\begin{aligned}
6n &\sum_{1\le k\le n/2} \frac{\Pr[c \text{ in } k]}{k}\Bigl(\log n - \log k + O(1)\Bigr) \\
&\le 6n \sum_{1\le k\le n/2} \frac{1}{k}\cdot\frac{1}{3n-2k+1}\left(1+\frac{k}{3n-2k+5}\right)\Bigl(\log n - \log k + O(1)\Bigr) \\
&\le 2 \sum_{1\le k\le n/2} \left(\frac{1}{k}+\frac{2}{3n-2k+5}\right)\left(1+\frac{k}{3n-2k+5}\right)\Bigl(\log n - \log k + O(1)\Bigr) \\
&= (\log n)^2 + O(\log n),
\end{aligned}
$$

where the main contribution comes from

$$
2 \sum_{1\le k\le n/2} \frac{\log n - \log k}{k} = (\log n)^2 + O(\log n).
$$

This proves (5.5) and thus completes the proof of the theorem. □

6. The Probability of a Single Cycle

We shall be concerned in this section with determining the probability

$$
q_{n/2} = \Pr[h = 1]
$$

that the fat-graph constructed from n triangles contains a single boundary cycle.

Lemma 6.1: We have

$$q_{2s} = 0$$

for $s \geq 0$.

Proof: Since $h = n/2 + \chi$ and χ is even, h has the same parity as $n/2$. Thus $\Pr[h = 1] = 0$ when $n/2$ is even. □

Our main result is the following.

Theorem 6.2: We have

$$q_{2s+1} = \frac{2^{4s+1}\, 3^{3s+1}\, (4s+2)!\, (6s+2)!\, (6s+3)!}{(s+1)!\, (3s+1)!\, (12s+6)!}$$

for $s \geq 0$.

Proof: Let $t = 2s+1 = n/2$. Then if $h = 1$, the single cycle has length $6t = 3n$. We employ a technique attributed by Bessis, Itzykson and Zuber [12] to J. M. Drouffe. This proof is uses the irreducible characters χ^α of the symmetric group S_{6t}, indexed by the partitions α of $6t$ (that is, non-decreasing sequences of positive integers summing to $6t$). The facts we need are found in the book by Sagan [46].

Let

$$Q_t = q_t\, (6t-1)!!$$

denote the number of pairings that result in a single cycle of length $6t$. Then

$$Q_t = \sum_\tau \delta_{[\tau],[2^{3t}]} \delta_{[\sigma\tau],[6t]}, \tag{6.1}$$

where the sum is over permutations τ in S_{6t} and σ is a fixed permutation of congugacy class $[3^{2t}]$ (that is, the conjugacy class of permutations containing $2t$ cycles of length 3). From the completeness relation for the characters of the symmetric group S_{6t} (see Theorem 1.10.3 in Sagan [46]), we have

$$\sum_\alpha \chi^\alpha(\phi)\, \chi^\alpha(\psi) = \frac{\delta_{[\phi],[\psi]}\, (6t)!}{\#[\phi]},$$

where the sum is over the partitions α of $6t$, ϕ and ψ are permutations, $[\phi]$ denotes the congugacy class containing ϕ, and $\#[\phi]$ denotes its cardinatlity. Substituting this relation in (6.1) yields

$$Q_t = \frac{\#[2^{3t}]\, \#[6t]}{(6t)!^2} \sum_{\alpha,\beta} \chi^\alpha([2^{3t}])\, \chi^\beta([6t]) \sum_\tau \chi^\alpha(\tau)\, \chi^\beta(\sigma\tau). \tag{6.2}$$

We shall need the formula

$$\sum_{\tau} \chi^{\alpha}(\tau)\, \chi^{\beta}(\sigma\tau) = \frac{\delta_{\alpha,\beta}\, \chi^{\alpha}(\sigma)\, (6t)!}{\chi^{\alpha}([1^{6t}])}. \tag{6.3}$$

This variant of the orthogonality relation for characters can be proved as follows. Let $\{a_{i,j}\}_{1\le i,j\le v}$ and $\{b_{p,q}\}_{1\le p,q\le w}$ be the matrix elements of the representations corresponding to χ^{α} and χ^{β}, respectively. From the proof of Theorem 1.9.3 in Sagan [46], we have

$$\sum_{\tau} a_{i,j}(\tau)\, b_{p,q}(\tau) = \frac{(6t)!\, \delta_{\alpha,\beta}\, \delta_{i,q}\, \delta_{j,p}}{\chi^{\alpha}([1^{6t}])},$$

since $v = \chi^{\alpha}([1^{6t}])$. (Note that $a_{i,j}(\tau^{-1}) = a_{i,j}(\tau)$, since every element is conjugate to its inverse in the symmetric group.) Setting $j = i$ and multiplying by $b_{q,p}(\sigma)$ yields

$$\sum_{\tau} a_{i,i}(\tau)\, b_{q,p}(\sigma)\, b_{p,q}(\tau) = \frac{(6t)!\, \delta_{\alpha,\beta}\, \delta_{i,q}\, \delta_{i,p}\, a_{q,p}(\sigma)}{\chi^{\alpha}([1^{6t}])}.$$

Summing over p yields

$$\sum_{\tau} a_{i,i}(\tau)\, b_{q,q}(\sigma\tau) = \frac{(6t)!\, \delta_{\alpha,\beta}\, \delta_{i,q}\, a_{q,i}(\sigma)}{\chi^{\alpha}([1^{6t}])};$$

summing over q yields

$$\sum_{\tau} a_{i,i}(\tau)\, \chi^{\beta}(\sigma\tau) = \frac{(6t)!\, \delta_{\alpha,\beta}\, a_{i,i}(\sigma)}{\chi^{\alpha}([1^{6t}])};$$

and summing over i yields (6.3). Substituting (6.3) in (6.2) yields

$$Q_t = \frac{\#[2^{3t}]\, \#[6t]}{(6t)!} \sum_{\alpha} \frac{\chi^{\alpha}([2^{3t}])\, \chi^{\alpha}([6t])\, \chi^{\alpha}([3^{2t}])}{\chi^{\alpha}([1^{6t}])}, \tag{6.4}$$

since $\sigma \in [3^{2t}]$.

To evaluate the sum in (6.4), we observe that

$$\chi^{\alpha}([6t]) = \begin{cases} (-1)^p, & \text{if } \alpha = [6t-p, 1^p]; \\ 0, & \text{otherwise.} \end{cases}$$

(This is Lemma 4.10.3 in Sagan [46].) Thus, setting $\lambda(p) = [6t-p, 1^p]$, we have

$$Q_t = \frac{\#[2^{3t}]\, \#[6t]}{(6t)!} \sum_{0\le p\le 6t} \frac{(-1)^p\, \chi^{\lambda(p)}([2^{3t}])\, \chi^{\lambda(p)}([3^{2t}])}{\chi^{\lambda(p)}([1^{6t}])}. \tag{6.5}$$

The denominator $\chi^{\lambda(p)}([1^{6t}])$ is the dimension of the representation coresponding to the partition $\lambda(p)$:

$$\chi^{\lambda(p)}([1^{6t}]) = \binom{6t-1}{p}.$$

(This follows from the Hook Formula, Theorem 3.10.2 in Sagan [46].) The characters in the numerator are given by

$$\chi^{\lambda(p)}([2^{3t}]) = (-1)^{p+\lfloor p/2 \rfloor} \binom{3t-1}{\lfloor p/2 \rfloor}$$

and

$$\chi^{\lambda(p)}([3^{2t}]) = (-1)^{p+\lfloor p/3 \rfloor} \binom{2t-1}{\lfloor p/3 \rfloor}.$$

(These follow from the Murnaghan-Nakayama Rule, Theorem 4.10.2 in Sagan [46].) Substituting these relations in (6.5) yields

$$Q_t = \frac{\#[2^{3t}]\,\#[6t]}{(6t)!} \sum_{0 \le p \le 6t} \frac{(-1)^{p+\lfloor p/2 \rfloor + \lfloor p/3 \rfloor} \binom{3t-1}{\lfloor p/2 \rfloor} \binom{2t-1}{\lfloor p/3 \rfloor}}{\binom{6t-1}{p}}.$$

Finally, we have $\#[2^{3t}] = (6t-1)!!$ and $\#[6t] = (6t-1)!$, which yields

$$Q_t = \frac{(6t-1)!!}{6t} \sum_{0 \le p \le 6t} \frac{(-1)^{p+\lfloor p/2 \rfloor + \lfloor p/3 \rfloor} \binom{3t-1}{\lfloor p/2 \rfloor} \binom{2t-1}{\lfloor p/3 \rfloor}}{\binom{6t-1}{p}}. \tag{6.6}$$

The sign pattern in (6.6) has a period of 12; this suggests that we aggregate the terms in groups of 6. Thus we make the substitution $q = \lfloor p/6 \rfloor$; the result is

$$Q_t = \frac{(6t-1)!!}{6t} \times \sum_{0 \le q \le t-1} \frac{(-1)^q \binom{3t-1}{3q} \binom{2t-1}{2q}}{\binom{6t-1}{6q}} \left[1 - \frac{4\,(6q+1)}{(6t-6q-1)} + \frac{(6q+1)(6q+5)}{(6t-6q-1)(6t-6q-5)}\right].$$

Making the substitution $t = 2s+1$, we obtain

$$Q_{2s+1} = \frac{(12s+5)!!}{12s+6} \times \sum_{0 \le q \le 2s} \frac{(-1)^q \binom{6s+2}{3q} \binom{4s+1}{2q}}{\binom{12s+5}{6q}} \left[1 - \frac{4\,(6q+1)}{(12s-6q+5)} + \frac{(6q+1)(6q+5)}{(12s-6q+5)(12s-6q+1)}\right].$$

Thus to prove the theorem, we must show that

$$T(s) = \sum_{0 \le q \le 2s} F(s,q) = 1, \tag{6.7}$$

where

$$F(s,q) = \frac{(-1)^q\,(1+36q^2+4s-72qs+24s^2)\binom{4s+1}{2q}\binom{6s+2}{3q}(s+1)!\,(3s+1)!\,(12s+5)!}{2^{4s}\,3^{3s}\,(1-6q+12s)\,(5-6q+12s)\binom{12s+5}{6q}(4s+2)!\,(6s+2)!\,(6s+3)!}.$$

Define

$$G(s,q) = F(s,q)\,R(s,q),$$

where

$$R(s,q) = \frac{A(s,q)}{B(s,q)},$$

$$\begin{aligned}A(s,q) = q\,(12s-6q+5)\,(12s-6q+1)\times &\\ \big(36q(1440s^5+5328s^4+6656s^3+2560s^2-734s-527)-&\\ 36q^2(1488s^4+4832s^3+5444s^2+2370s+271)-&\\ 1296q^4(2s+1)\,(s+1)+10368q^q(2s+1)\,(s+1)^2-&\\ 13824s^6-47232s^5-27984s^4+76288s^3+75909s+15050\big)&\end{aligned}$$

and

$$\begin{aligned}B(s,q) = 2^4\,3^2\,(2s-q+1)\,(2s-q+2)\,(s+1)\times &\\ (2s+3)^2\,(6s+5)\,(6s+7)\,(24s^2-72qs+4s+36q^2+1).&\end{aligned}$$

Then we have

$$F(s+1,q) - F(s,q) = G(s,q+1) - G(s,q).$$

Summing this result over all q yields

$$T(s+1) - T(s) = 0,$$

since $F(s,q)$ vanishes outside the range of summation in (6.7). Since $T(0) = 1$,we obtain (6.7) for all $s \ge 0$ by induction. □

Corollary 6.3: We have

$$\Pr[h=1] = \begin{cases} 0, & \text{if } n/2 \text{ even;} \\ \frac{2}{3n} + O\left(\frac{1}{n^2}\right), & \text{if } n/2 \text{ odd.} \end{cases}$$

Proof: Applying Stirling's asymptotic formula in the form

$$n! = \frac{(2\pi n)^{1/2} n^n}{e^n} \left(1 + O\left(\frac{1}{n}\right)\right)$$

to Theorem 6.2 yields

$$q_{2s+1} = \frac{1}{6s} + O\left(\frac{1}{s^2}\right).$$

This relation together with Lemma 6.1 yields the corollary. □

7. The Classification of Cycles

In this section we shall introduce a classification of boundary cycles in the fat graph model, based on their "self-interactions". In this classification, each cycle is either "simple" or "complex", and each simple cycle is of some finite order. We shall give a heuristic argument that predicts the expected number of simple cycles of each finite order. In the following sections, we shall give rigourous confirmations of these predictions for the two lowest orders.

Let C be a boundary cycle in a fat graph. The *self-interaction surface* $S(C)$ of C is the union of all ribbons traversed by C in both directions (that is, all ribbons with both links marked), together with all triangles incident with these ribbons (that is, all triangles visited more than once by C).

If $S(C)$ contains two ribbons incident with a given triangle, then it also contains the third ribbon incident with that triangle. Thus every triangle in $S(C)$ is incident with either one ribbon or three. Since every ribbon is incident with two triangles, it follows that the number of triangles in a connected component of $S(C)$ is even.

We shall say that a connected component of $S(C)$ is *simple* if it is simply connected, and that it is *complex* otherwise. A simple cycle will be said to have *order* b if every connected component of $S(C)$ contains at most $2b$ triangles. Thus a cycle has order zero if it traverses no ribbon in both directions (or equivalently, if it visits at most one corner of any triangle); it has order one if it visits at most two corners in any triangle.

The top pairing in Figure 1 yields two simple cycles of order zero and one of order one; the middle pairing yields three simple cycles of order zero; and the bottom pairing yields complex cycle.

Let us now consider, in a heuristic way, the expected number of simple cycles of various finite orders. Let the random variable s_b denote the number of simple cycles of order at

most b. We start with cycles of order zero. A cycle cannot have a doubly traversed ribbon unless it has length at least four, and it *must* have a doubly traversed ribbon if its length exceeds n. Thus we expect most short cycles to have order zero, and most long cycles to have higher order. Consideration of the "birthday effect" (if there are n days in a year, then among $n^{1/2}$ people in a room there is a significant probability that two have the same birthday) leads one to anticipate that the transition will occur for cycles of length around $n^{1/2}$. From the results of Section 4, we have that the expected number of cycles of length k is about $1/k$. Thus we anticipate that $\mathrm{Ex}[s_0] \approx \sum_{1\le k\le n^{1/2}} \frac{1}{k} = \frac{1}{2}\log n + O(1)$. We can refine this estimate in the following way. Although all doubly traversed ribbons are excluded from cycles of order zero, we shall focus our attention on "minimal forbidden self-interactions", which are isolated doubly traversed ribbons (that is, components of $S(C)$ containing exactly two triangles joined by a single edge). We assume the number f_0 of isolated doubly traversed ribbons is approximately Poisson distributed, so that $\Pr[f_0 = 0] \approx \exp -\mathrm{Ex}[f_0]$. We shall see in the next paragraph that we have $\mathrm{Ex}[f_0] \approx k^2/6n$. This leads us to anticipate that

$$\begin{aligned}
\mathrm{Ex}[s_0] &\approx \sum_{1\le k\le n} \frac{1}{k}\exp(-k^2/6n) \\
&\approx \frac{1}{2}\log(6n) + \frac{\gamma}{2}.
\end{aligned}$$

In Section 8 we shall confirm this prediction in the form

$$\mathrm{Ex}[s_0] = \frac{1}{2}\log(6n) + \frac{\gamma}{2} + O\left(\frac{\log n}{n^{1/3}}\right).$$

Let us now justify the approximation $\mathrm{Ex}[f_0] \approx k^2/6n$. We consider how the algorithm of Section 4 can doubly traverse a ribbon between the arc (c_1^-, c_1) of triangle t_1 and the arc (c_2^-, c_2) of triangle t_2 (without doubly traversing any other ribbon incident with t_1 or t_2). This can happen if corner c_1^+ is drawn from the urn at some draw d_1, then corner c_2 is drawn at the immediately following draw d_1+1, and finally corner c_2^+ is drawn at some subsequent draw $d_2 > d_1+1$. The two triangles t_1 and t_2 and their corners c_1 and c_2 can be chosen in about $9n^2$ ways. The two draws d_1 and d_2 can be chosen in about $k^2/2$ ways. Each of the three draws occurs with probability about $1/3n$, for an overall probability of about $1/27n^3$. The product of these factors must first be multiplied by a factor of 2, since the same ribbon is doubly traversed (with the traversals occurring in the opposite order) if c_2^+ is drawn at d_1, c_1 is drawn at d_1+1 and c_1^+ is drawn at d_2. Finally, we must divide

by a factor of 2, since the same ribbon is doubly traversed if c_1 of t_1 is exchanged with c_2 of t_2. The product of all these factors is $k^2/6n$.

We now turn to cycles of order at most one. Here the minimal forbidden self-interaction is an isolated triply visited triangle, and reasoning by analogy with the birthday effect we anticipate that these will begin to appear when k is about $n^{2/3}$. This gives the estimate $\mathrm{Ex}[s_1] \approx \sum_{1 \le k \le n^{2/3}} \frac{1}{k} = \frac{2}{3}\log n + O(1)$. Again we can refine this result by estimating the expectation of the number f_1 of isolated triply visited triangles. In the following paragraph we shall argue that $\mathrm{Ex}[f_1] \approx k^3/27n^2$, which leads us to anticipate that

$$\begin{aligned}\mathrm{Ex}[s_1] &\approx \sum_{1 \le k \le 2n} \frac{1}{k}\exp(-k^3/27n^2) \\ &\approx \frac{2}{3}\log n + \log 3 + \frac{2\gamma}{3}.\end{aligned}$$

In Section 9 we shall confirm this prediction in the form

$$\mathrm{Ex}[s_1] = \frac{2}{3}\log n + \log 3 + \frac{2\gamma}{3} + O\left(\frac{(\log n)^7}{n^{1/8}}\right).$$

Let us now justify the approximation $\mathrm{Ex}[f_1] \approx k^3/27n^2$. We consider how the algorithm of Section 4 can triply visit triangle t_4 by doubly traversing the ribbon joining its arc (c_4^-, c_4) to the arc (c_1^-, c_1) of a triangle t_1, the ribbon joining its arc (c_4^+, c_4^-) to the arc (c_2^-, c_2) of a triangle t_2, and the ribbon joining its arc (c_4, c_4^+) to the arc (c_3^-, c_3) of a triangle t_3 (without doubly traversing any of the other ribbons incident with t_1, t_2 or t_3). This can happen if corner c_1^+ is drawn from the urn at some draw d_1, corner c_4 is drawn at the immediately following draw $d_1 + 1$, corner c_2 is drawn at the immediately following draw $d_1 + 2$, then corner c_2^+ is drawn at some subsequent draw $d_2 > d_1 + 2$ and corner c_3 is drawn at the immediately following draw $d_2 + 1$, and finally corner c_3^+ is drawn at some subsequent draw $d_3 > d_2 + 1$.

The four triangles t_1, t_2, t_3 and t_4 and their corners c_1, c_2, c_3 and c_4 can be chosen in about $81n^4$ ways. The three draws d_1, d_2 and d_3 can be chosen in about $k^3/6$ ways. Each of the six draws occurs with probability about $1/3n$, for an overall probability of about $1/729n^6$. The product of these factors must first be multiplied by a factor of 6, since the same triangle is triply visited (with the visits occurring in a different order) if the triangles t_1, t_2 and t_3 are visited in any of the five other permutations of order considered above. Finally, we must divide by a factor of 3, since the same triangle is triply visited if c_1 of t_1, c_2 of t_2 and c_3 of t_3 are cyclically permuted. The product of all these factors is $k^3/27n^2$.

We now turn to cycles of order at most $b \geq 2$. If the minimal forbidden self-interactions in the fat graph are shrunk to parts of the corresponding thin graph, the results are trees with $2b+2$ vertices ($b+2$ leaves and b internal vertices) and $2b+1$ edges. These trees are cubic plane trees, since every internal vertex has degree 3 and the trees are regarded as embedded in the plane, so that two trees are regarded as isomorphic if there is a bijection between their sets of vertices that preserves adjacency and also preserves the cyclic order of the neighbours around each internal vertex. (For each $b \in \{0, 1, 2, 3\}$, there is just one such tree, up to isomorphism; for each $b \geq 4$ there are more than one.)

We need to estimate the expectation of the number f_b of occurrences of such minimal forbidden self-interactions. Consider a cubic plane tree with $2b$ vertices. There are about $(3n)^{2b+2}$ ways of choosing $2b+2$ triangles and one corner from each triangle. The forbidden self-interaction will occur if $3b+3$ specific corners are drawn in $b+2$ series of draws. The probability of these corners being drawn is about $1/(3n)^{3b+3}$, and the number of ways of choosing the $b+2$ draws that begin the successive series is about $k^{b+2}/(b+2)!$. We must multiply by a factor of $(b+2)!$ accounting for the order in which the visits to the self-interaction at the draws that begin the successive series occur, and we must divide by a factor of 1, 2 or 3 accounting for the automorphisms of the tree. (A plane cubic tree can be symmetric about a vertex, with three automorphisms, or symmetric about an edge, with two automorphisms, or rigid, with just the trivial automorphism.) Thus we have

$$\mathrm{Ex}[f_b] \approx \frac{k^{b+2}\, T_b}{(3n)^{b+1}}, \tag{7.1}$$

where T_b is the sum over all cubic plane trees with $2b+2$ vertices of the weight $1/q$ for each tree with q automorphisms.

The eight trees corrresponding to minimal forbidden self-interactions for $0 \leq b \leq 4$, together with their weights, are shown in Figure 3. The four trees on the left arise for $0 \leq b \leq 3$; the four on the right arise for $b = 4$.

Proposition 7.1: The sum of weights over plane cubic trees containing $2b+2$ vertices is

$$T_b = \frac{1}{(b+1)(b+2)} \binom{2b}{b}.$$

Proof: We shall consider *rooted* trees, in which one vertex is distinguished as the root. We begin by considering cubic plane trees that are rooted at a leaf. Let the generating function for such trees, in which the number of leaf-rooted cubic plane trees with a vertices is the coefficient of x^a, be $F(x)$. Such a tree either contains just two adjacent vertices, or

it can be constructed by identifying the two roots of two such trees to form the neighbour of a new leaf, which is the root of the constructed tree. Thus we have $F(x) = x^2 + F(x)^2$, which implies

$$F(x) = \frac{1 - (1 - 4x^2)^{1/2}}{2}.$$

Let $\Phi_3(x)$ be the generating function for (unrooted) cubic plane trees with three automorphisms. Such a tree can be constructed by identifying the roots of three copies of a leaf-rooted cubic plane tree to form an internal vertex (the vertex of symmetry) of the constructed tree. Thus we have $\Phi_3(x) = F(x^3)/x^2$, which implies

$$\Phi_3(x) = \frac{1 - (1 - 4x^6)^{1/2}}{2x^2}.$$

Let $\Phi_2(x)$ be the generating function for (unrooted) cubic plane trees with two automorphisms. Such a tree can be constructed by identifying edges incident with the roots of two copies of a leaf-rooted cubic plane tree to form an edge (the edge of symmetry) of the constructed tree. Thus we have $\Phi_2(x) = F(x^2)/x^2$, which implies

$$\Phi_2(x) = \frac{1 - (1 - 4x^4)^{1/2}}{2x^2}.$$

Let $G(x)$ be the generating function for rooted cubic plane trees (where now the root may be any vertex, either leaf or internal). Such a tree is either a leaf-rooted cubic plane tree, or it can be constructed by identifying the roots of three leaf-rooted cubic plane trees to form an internal vertex of the constructed tree. This procedure constructs each internally rooted cubic plane tree three times, unless the three trees that are combined are isomorphic, in which case the resulting tree, which has three automorphisms, is constructed once. Thus we have $G(x) = F(x) + F(x)^3/3x^2 + 2F(x^3)/3x^2$, which implies

$$G(x) = \frac{3 - (1 + 2x^2)(1 - 4x^2)^{1/2} - 2(1 - 4x^6)^{1/2}}{6x^2}.$$

Let $H_1(x)$ be the generating function for rooted rigid cubic plane trees (where the root may be any vertex, and the trees have only the trivial automorphism). The number of rooted cubic plane trees with two automorphisms is given by the generating function

$$H_2(x) = y \frac{d}{dy} \frac{F(y)}{y} \bigg|_{y=x^2} = \frac{1 - (1 - 4x^2)^{1/2}}{2x^2(1 - 4x^2)^{1/2}},$$

and the number of rooted cubic plane trees with three automorphisms is given by the generating function

$$H_3(x) = x\frac{d}{dy}F(y)\bigg|_{y=x^3} = \frac{2x^4}{(1-4x^6)^{1/2}}.$$

Thus we have

$$\begin{aligned} H(x) &= G(x) - H_2(x) - H_3(x) \\ &= \frac{6-(1+2x^2)(1-4x^2)^{1/2}}{6x^2} - \frac{1}{2x^2(1-4x^4)} - \frac{1+2x^6}{3x^2(1-4x^6)^{1/2}}. \end{aligned}$$

Let $\Phi_1(x)$ be the generating function for (unrooted) rigid cubic plane trees. We have

$$\begin{aligned} \Phi_1(x) &= \int_0^x \frac{H_1(y)\,dy}{y} \\ &= \frac{(1-4x^2)^{3/2} + 3(1-4x^4)^{1/2} + 2(1-4x^6)^{1/2} - 6}{12x^2}. \end{aligned}$$

The generating function $\Psi(x)$ for the numbers T_b is then given by

$$\begin{aligned} \Psi(x) &= \left(\Phi_1(y) + \frac{1}{2}\Phi_2(y)x + \frac{1}{3}\Phi_3(y)\right)\bigg|_{y=x^{1/2}} \\ &= \frac{(1-4x)^{3/2} - (1-6x)}{12x}. \end{aligned}$$

The proposition now follows using the binomial theorem. □

The numbers $6T_b$ (which are integers) occur as a generalization of the Catalan numbers in the work of Gessel [47], who asked for an enumerative interpretation of them. Such an enumerative interpretation is provided by considering drawings of cubic plane trees in which every edge is drawn at an angle that is a multiple of $\pi/3$ from a reference line. Cubic plane trees themselves are equivalent to "flexagons", which have been counted by Oakley and Wisner [48].

Substituting the result of Proposition 7.1 into (7.1) yields

$$\mathrm{Ex}[f_b] \approx \frac{k^{b+2}}{(3n)^{b+1}} \cdot \frac{1}{(b+1)(b+2)}\binom{2b}{b}.$$

Thus we anticipate that

$$\begin{aligned} \mathrm{Ex}[s_b] &\approx \sum_{1\le k\le 3n} \frac{1}{k}\exp - \left(\frac{k^{b+2}}{(3n)^{b+1}} \cdot \frac{1}{(b+1)(b+2)}\binom{2b}{b}\right) \\ &\approx \frac{b+1}{b+2}\log(3n) + \frac{(b+1)\gamma}{b+2} - \frac{1}{b+2}\log\left(\frac{1}{(b+1)(b+2)}\binom{2b}{b}\right). \end{aligned}$$

We note that

$$\frac{1}{b+2}\log\left(\frac{1}{(b+1)(b+2)}\binom{2b}{b}\right) \to \log 4$$

as $b \to \infty$. This suggests the conjecture that the number of complex cycles is asymptotic to $\log 4 = 1.386\ldots$.

8. Cycles of Order Zero

A simple cycle of order zero has length at most n (since each of n triangles is visited at most once), and for $1 \le k \le n$ the expected number of such cycles of length k is

$$\frac{1}{k} \cdot \frac{3^k\, n!}{(n-k)!} \cdot \frac{(3n-2k-1)!!}{(3n-1)!!}.$$

(The middle factor counts the ways of choosing an ordered list of k corners, no two on the same triangle. The last factor gives the probability of connecting these corners into a cycle in the order given. This counts each cycle k times, hence the first factor.) The expected number s_0 of simple cycles of order zero is thus

$$\mathrm{Ex}[s_0] = \sum_{1\le k\le n} \frac{1}{k} \cdot \frac{3^k\, n!}{(n-k)!} \cdot \frac{(3n-2k-1)!!}{(3n-1)!!}.$$

Theorem 8.1: We have

$$\mathrm{Ex}[s] = \frac{1}{2}\log(6n) + \frac{\gamma}{2} + O\left(\frac{\log n}{n^{1/3}}\right).$$

Proof: If we define

$$\begin{aligned} F_k &= \frac{\prod_{1\le j\le k} (3n-3j+3)}{\prod_{1\le j\le k} (3n-2j+1)} \\ &= \frac{\prod_{1\le j\le k} \left(1 - \frac{j-1}{n}\right)}{\prod_{1\le j\le k} \left(1 - \frac{2j-1}{3n}\right)}, \end{aligned}$$

we can write

$$\mathrm{Ex}[s] = \sum_{1\le k\le n} \frac{1}{k} F_k.$$

For the numerator of F_k we have

$$\prod_{1\le j\le k}\left(1-\frac{j-1}{n}\right) = \exp \sum_{1\le j\le k} \log\left(1-\frac{j}{n}+O\left(\frac{1}{n}\right)\right)$$
$$= \exp \sum_{1\le j\le k} \left(-\frac{j}{n}+O\left(\frac{1}{n}\right)+O\left(\frac{j^2}{n^2}\right)\right)$$
$$= \exp \left(-\frac{k^2}{2n}+O\left(\frac{k}{n}\right)+O\left(\frac{k^3}{n^2}\right)\right).$$

Similar estimation of the denominator yields

$$\prod_{1\le j\le k}\left(1-\frac{2j-1}{3n}\right) = \exp \left(-\frac{k^2}{3n}+O\left(\frac{k}{n}\right)+O\left(\frac{k^3}{n^2}\right)\right).$$

Thus we have

$$F_k = \exp \left(-\frac{k^2}{6n}+O\left(\frac{k}{n}\right)+O\left(\frac{k^3}{n^2}\right)\right).$$

Define

$$l = \lceil (6n\log n)^{1/2}\rceil.$$

Then we have

$$F_l = O\left(\frac{1}{n}\right).$$

Since F_k is a decreasing function of k, we have

$$\sum_{l<k\le n} \frac{1}{k}\cdot F_k = O\left(\frac{1}{n}\sum_{l<k\le n}\frac{1}{k}\right)$$
$$= O\left(\frac{1}{l}\right)$$
$$= O\left(\frac{1}{n^{1/3}}\right).$$

Thus to prove the theorem it will suffice to show that

$$\sum_{1\le k\le l} \frac{1}{k}\cdot F_k = \frac{1}{2}\log(6n)+\frac{\gamma}{2}+O\left(\frac{\log n}{n^{1/3}}\right). \tag{8.1}$$

For $k \le l$ we have

$$F_k = \exp \left(-\frac{k^2}{6n}+O\left(\frac{(\log n)^{3/2}}{n^{1/2}}\right)\right)$$
$$= \left(1+O\left(\frac{(\log n)^{3/2}}{n^{1/2}}\right)\right)\exp\left(-\frac{k^2}{6n}\right),$$

so that

$$\sum_{1\le k\le l} \frac{1}{k}\cdot F_k = \left(1+O\left(\frac{(\log n)^{3/2}}{n^{1/2}}\right)\right) \sum_{1\le k\le l} \frac{1}{k}\cdot \exp\left(-\frac{k^2}{6n}\right).$$

Thus to prove (8.1) it will suffice to show that

$$\sum_{1\le k\le l} \frac{1}{k}\cdot \exp\left(-\frac{k^2}{6n}\right) = \frac{1}{2}\log(6n) + \frac{\gamma}{2} + O\left(\frac{\log n}{n^{1/3}}\right). \tag{8.2}$$

Define

$$m = \lceil (6n)^{1/3} \rceil.$$

Then for $k \le m$ we have

$$\begin{aligned} F_k &= \exp\ O\left(\frac{1}{n^{1/3}}\right) \\ &= 1 + O\left(\frac{1}{n^{1/3}}\right). \end{aligned}$$

This yields

$$\begin{aligned} \sum_{1\le k\le m} \frac{1}{k}\cdot \exp\left(-\frac{k^2}{6n}\right) &= \left(1+O\left(\frac{1}{n^{1/3}}\right)\right) \sum_{1\le k\le m} \frac{1}{k} \\ &= \left(1+O\left(\frac{1}{n^{1/3}}\right)\right) \left(\frac{1}{3}\log(6n) + \gamma + O\left(\frac{1}{n^{1/3}}\right)\right), \end{aligned}$$

where $\gamma = 0.5772\ldots$ is Euler's constant. Thus to prove (8.2) it will suffice to show that

$$\sum_{m<k\le l} \frac{1}{k}\cdot \exp\left(-\frac{k^2}{6n}\right) = \frac{1}{6}\log(6n) - \frac{\gamma}{2} + O\left(\frac{\log n}{n^{1/3}}\right). \tag{8.3}$$

We have

$$\sum_{m<k\le l} \frac{1}{k}\cdot \exp\left(-\frac{k^2}{6n}\right) = \int_m^l \frac{1}{z}\cdot \exp\left(-\frac{z^2}{6n}\right)\, dz + O\left(\frac{1}{n^{1/3}}\right),$$

since we may bound the difference between a sum and an integral by the total variation of the integrand. The substitution $z = (6ny)^{1/2}$ yields

$$\int_m^l \frac{1}{z}\cdot \exp\left(-\frac{z^2}{6n}\right)\, dz = \frac{1}{2}\int_{m^2/6n}^{l^2/6n} \frac{1}{y}\cdot \exp(-y)\, dy.$$

Since

$$\int_{l^2/6n}^{\infty} \frac{1}{y} \cdot \exp(-y)\, dy = O\left(\frac{6n}{l^2} \cdot \exp\left(-\frac{l^2}{6n}\right)\right)$$
$$= O\left(\frac{\log n}{n}\right),$$

we may raise the upper bound of the integral from $l^2/6n$ to ∞, and thus obtain an expression in terms of the exponential integral:

$$\frac{1}{2}\int_{m^2/6n}^{l^2/6n} \frac{1}{y} \cdot \exp(-y)\, dy = \frac{1}{2}\int_{m^2/6n}^{\infty} \frac{1}{y} \cdot \exp(-y)\, dy + O\left(\frac{\log n}{n}\right)$$
$$= -\frac{1}{2}\mathrm{Ei}\left(-\frac{m^2}{6n}\right) + O\left(\frac{\log n}{n}\right),$$

where

$$-\mathrm{Ei}(-x) = \int_{x}^{\infty} \frac{1}{w} \cdot \exp(-w)\, dw$$

(see Lebedev [49], §1.3). Using the asymptotic expansion

$$-\mathrm{Ei}(-x) = \log\frac{1}{x} - \gamma + O(x),$$

we obtain

$$-\frac{1}{2}\mathrm{Ei}\left(-\frac{m^2}{6n}\right) = \frac{1}{2}\log\frac{6n}{m^2} - \frac{\gamma}{2} + O\left(\frac{m^2}{6n}\right)$$
$$= \frac{1}{6}\log(6n) - \frac{\gamma}{2} + O\left(\frac{1}{n^{1/3}}\right).$$

Combining these results yields (8.3), and thus completes the proof of the theorem. □

9. Cycles of Order One

We begin by deriving an exact formula for the expected number of simple cycles of order one.

Proposition 9.1: We have

$$\mathrm{Ex}[s_1] = \sum_{1\le k\le 2n} \sum_{0\le j\le k/4} \frac{1}{k}\left[\binom{k-2j}{2j} + \binom{k-2j-1}{2j-1}\right](2j-1)!! \times$$
$$\frac{3^{k-2j}\, n!}{(n-k+2j)!} \cdot \frac{(3n-k+j-1)!!}{(3n-1)!!}. \tag{9.1}$$

Proof: If a cycle visits each of the n triangles at most twice, it can have length at most $2n$. Furthermore, if a cycle visits each triangle at most twice, then each traversal of a doubly

traversed ribbon must be immediately followed by the traversal of a singly traversed ribbon, and thus such a cycle of length k can have at most $k/4$ doubly traversed ribbons. Thus it remains to show that the summand in (9.1) is the expected number of cycles of length k with j doubly traversed ribbons.

Consider a cyclic directed graph $C = (V, E)$ with vertices $V = \{v_1, \ldots, v_k\}$ and edges $E = \{(v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, v_1)\}$. A *template* $T = (I, J)$ comprises a set $I \subseteq E$ of $2j$ edges of C, subject to the condition that no two edges in I are consecutive, together with a partition J of I into j pairs of edges. The number of templates is

$$\left[\binom{k-2j}{2j} + \binom{k-2j-1}{2j-1}\right] (2j-1)!!,$$

since there are $\binom{k-2j}{2j} + \binom{k-2j-1}{2j-1}$ ways of choosing the $2j$ edges in I (there are $\binom{k-2j}{2j}$ ways that exclude the edge (v_k, v_1), and $\binom{k-2j-1}{2j-1}$ ways that include it), and $(2j-1)!!$ ways to partition these edges to form the j pairs in J. (We agree that $(-1)!! = 1$, as a special case of the formula $(2j-1)!! = (2j)!/2^j\, j!$.)

The cyclic group of order k acts on C in an obvious way, and this induces an action on the set of templates. Let $p(T)$ denote the number of templates in the orbit of T under this action, and let $q(T)$ denote the order of the automorphism group of T. Then we have $p(T)\, q(T) = k$.

Next consider a boundary cycle in the fat graph with length k and j doubly traversed ribbons. We shall construct a template as follows. Pick a corner c_1 on the cycle, and then define $c_2, \ldots, c_k$ to be the the successive corners of the cycle. For each ribbon that is doubly traversed by the links (c_f, c_{f+1}) and (c_g, c_{g+1}) of the cycle, put the edges (v_f, v_{f+1}) and (v_g, v_{g+1}) into I and put the pair $\{(v_f, v_{f+1}), (v_g, v_{g+1})\}$ into J. Different templates can be obtained from the same boundary cycle by different choices of the corner c_1; $p(T)$ different templates are obtained in this way, since if we advance the choice of c_1 by $p(T)$ corners we obtain the same template.

Now consider the boundary cycles corresponding to a given template T. The cycle visits k corners on $k - 2j$ triangles, but once the first visited corner on a triangle is chosen, the remaining visited corner on that triangle is determined. The triangles can be chosen in $n!/(n-k+2j)!$ ways, and the first visited corner on each triangle can be chosen in 3^{k-2j} ways. Different choices of triangles and corners can produce the same boundary cycle; each boundary cycle is produced $q(T)$ times, since if we advance the choices of vertices $k/q(T) = p(T)$ positions we produce the same cycle. For such a cycle to occur in the fat graph, $k - j$ specific draws must occur, corresponding to the first traversals

of the $k-j$ ribbons traversed by the cycle. The probability of these draws occurring is $(3n-k+j-1)!!/(3n-1)!!$.

Finally, the expected number of boundary cycles of length k with j doubly traversed ribbons is

$$\sum_T \frac{1}{p(T)} \cdot \frac{3^{k-2j}\, n!}{q(T)\,(n-k+2j)!} \cdot \frac{(3n-k+j-1)!!}{(3n-1)!!},$$

where the sum is over all templates T. Since $p(T)\,q(T) = k$, we obtain the summand in (9.1). □

In principle, the asymptotic behaviour of $\mathrm{Ex}[s_1]$ can be obtained by direct analysis (9.1). We have found it most convenient, however, to bound the terms with the smallest and largest values of k through separate arguments, and thus to analyze (9.1) only for a set of intermediate values that includes the transition (around $n^{2/3}$) from significant terms to negligible ones. We give these separate arguments in the following two propositions.

Proposition 9.2: Let $m = n^{5/8}$. The expected number of cycles of order at most one and length at most m is

$$\log m + \gamma + O\left(\frac{1}{n^{1/8}}\right).$$

Proof: From the estimates in the proof of Theorem 4.1, we have that the expected number of cycles of length at most m is

$$\sum_{1\le k\le m} \left(\frac{1}{k} + O\left(\frac{1}{n}\right)\right) = \log m + \gamma + O\left(\frac{1}{n^{3/8}}\right).$$

The obstruction to a cycle having order at most one is a triply visited triangle. Thus it will suffice to show that the expected number of cycles of length at most m that contain a triply visited triangle is $O(1/n^{1/8})$. This expectation is in turn at most the expected number of triply visited triangles that are created during the first m iterations of the algorithm of Section 4.

There are three ways in which a triply visited triangle can occur: the three ribbons incident with it can be incident with three other triangles, or with just two, or with just one. The expected number of triply visited triangles for which the ribbons are incident with three other triangles was dealt with heuristically in Section 6. This argument can be made rigourous, at the cost of lowering our sights to an upper bound that holds within a constant factor. There are $O(n^4)$ ways of choosing corners c_1, c_2, c_3 and c_4 of triangles t_1, t_2, t_3 and t_4; there are $O(m^3)$ ways of choosing three draws d_1, d_2 and d_3; and the probability that the corners drawn at draws d_1, d_1+1, d_1+2, d_2, d_2+1 and d_3 create

a triply visited triangle is $O(1/n^6)$. Thus this expectation is $O(m^3/n^2) = O(1/n^{1/8})$. A similar argument shows that the case in which the three ribbons incident with the triply visited triangle are incident with two other triangles gives a smaller contribution, $O(n^4 k^2/n^6) = O(m^2/n^2) = O(1/n^{3/4})$. Finally, the case in which the three ribbons are all incident with one other triangle (which can happen only if $k = 6$) gives a still smaller contribution, $O(n^2/n^3) = O(1/n)$. □

For the next proposition, we shall need the following lemma.

Lemma 9.3: Let X denote the number of successes among h trials that each succeed independently with probability at most p. Then

$$\Pr[X > 2hp] \leq (e/4)^{hp}.$$

If on the other hand the trials succeed with probability at least q, then

$$\Pr[X < hq/2] \leq (2/e)^{hq/2}.$$

Proof: For the first inequality we may assume that the trials succeed with probability exactly p, since the resulting random variable is majorized by X. We may also assume that $p < 1/2$, since otherwise $\Pr[X > 2hp] = 0$. If

$$Y = \begin{cases} 0, & \text{if } X \leq 2hp, \\ 1, & \text{if } X > 2hp, \end{cases}$$

then $\Pr[X > 2hp] = \mathrm{Ex}[Y]$. If $Z = T^{X-2hp}$ (where $T > 1$ is a parameter to be chosen later), then $Y \leq Z$ and so $\mathrm{Ex}[Y] \leq \mathrm{Ex}[Z]$. Thus it will suffice to estimate $\mathrm{Ex}[Z]$. Since X is the sum of h independent random variables that assume the value 1 with probability p and the value 0 with probability $1 - p$, T^X is the product of h independent random variables that have expected value $pT + 1 - p$. Thus

$$\mathrm{Ex}[Z] = (pT + 1 - p)^h \, T^{-2hp}.$$

Choosing $T = 2(1-p)/(1-2p)$ and using the inequality $1+x \leq e^x$ yields the first inequality of the lemma. A similar argument yields the second inequality. □

Lemma 9.3 is an instance of Chernoff's inequality.

Proposition 9.4: Let $l = \lceil n^{2/3} \log n \rceil$. Then the expected number of cycles of order at most one with length exceeding $12l$ is $O(1/n)$.

Proof: From the proof of Theorem 4.1, we have that the expected number of cycles of order at most one with length exceeding $12l$ is

$$3n \sum_{k>12l} \frac{\Pr[c_0 \text{ in a } k\text{-cycle of order at most one}]}{k} \\ \leq (n/4l) \Pr[c_0 \text{ in a cycle of order at most one and length exceeding } 12l].$$

Thus it will suffice to show that

$$\Pr[c_0 \text{ in a cycle of order at most one at length at least } 12l] = O\left(\frac{1}{n^2}\right).$$

This probability is at most the probability that the algorithm of Section 4 runs for $6l$ iterations without creating a triply visited triangle.

To estimate this probabilty, we shall use an analysis that assigns colours to the corners in the urn during the execution of the algorithm. Corners will initially be white, but may subsequently be recoloured red or blue. The rules for colouring will be such that if a blue corner is drawn from the urn during the first $6l$ iterations, then either the cycle containing c_0 has length less than $12l$ or it contains a triply visited triangle. Thus it will suffice to show that the probability that no blue corner is drawn during the first $6l$ iterations is $O(1/n)$.

The rules for colouring are as follows. Initially, all corners are white except for c_0^-, which is red, and c_0^+, which is blue. If ever a blue corner is drawn, we stop the analysis. Whenever a white corner c is drawn, then if the immediately preceding corner drawn was white, we colour c^+ red, but if the immediately preceding corner drawn was red, we colour c^+ blue. It is easy to verify that drawing a blue corner either closes the cycle (which then has length at most $12l$) or creates a triply visited triangle. (It is possible to create a triply visited triangle without drawing a blue corner; the rules given have been chosen to be as simple as possible while yielding the desired upper bound.)

The rules specify how corners in the urn become coloured. There are two ways in which coloured corners can leave the urn: they can be drawn or they can be removed from the urn after becoming the *head* of the marked path. But at any given iteration, a corner c can be removed in this way only if one particular corner d (such that d^- is the rear of the unmarked path whose front is c) is drawn. Thus the probability of removing a coloured corner from the urn is never greater than the probability of drawing it.

We shall now obtain estimates for the numbers of corners of various colours in the urn at various times during the first $6l$ iterations. We shall divide these iterations into three *phases* (phases I, II and III), each comprising $2l$ iterations.

At any time during these $6l$ iterations, at most $12l$ corners have been drawn or removed from the urn, and thus the urn always contains at most $3n$ and at least $3n - 12l \geq 3n/2$ corners. Since at most one corner is coloured per iteration, there are always at least $(3n - 12l) - 6l = 3n - 18l$ white corners in the urn, and thus the probability of drawing a white corner is always at least $(3n - 18l)/3n \geq 1/2$.

We start by considering the number of corners coloured red during phase I. A corner is coloured red whenever two consecutive draws yield white corners. For each of l disjoint pairs of consecutive draws, this probability is at least $(1/2)(1/2) = 1/4$. Using Lemma 9.3 with $h = l$ and $q = 1/4$, we have that, except with probability at most $(2/e)^{l/8}$, at least $l/8$ corners are coloured red during phase I.

Next we consider the number of red corners drawn or removed during phases I and II. During these $4l$ iterations, there are never more than $4l$ red corners in the urn. Thus the probability that a red corner is drawn or removed in any iteration is at most $8l/3n$. Using Lemma 9.3 with $h = 4l$ and $p = 8l/3n$, we have that, except with probability at most $(e/4)^{32l^2/3n}$, at most $64l^2/3n$ red corners are drawn or removed during phases I and II. Thus, except with probability at most

$$(2/e)^{l/8} + (e/4)^{32l^2/3n},$$

there are at least $l/8 - 64l^2/3n \geq l/16$ red corners in the urn throughout phase II.

Next we consider the number of corners coloured blue during phase II. A corner is coloured blue whenever two consecutive draws yield a red corner followed by a white corner. For each of l disjoint pairs of consecutive draws, this probability is at least $\big((l/16)/3n\big)(1/2) = l/96n$. Using Lemma 9.3 with $h = l$ and $q = l/96n$, we have that, except with probability at most $(2/e)^{l^2/192n}$, at least $l^2/192n$ corners are coloured blue during phase II.

We shall also need an upper bound, holding with high probability, for the number of corners coloured blue during phases I, II and III. For each of $6l$ draws, the probability of colouring a corner blue is at most $\big(6l/(3n/2)\big) = 4l/n$. Using Lemma 9.3 with $h = 6l$ and $p = 4l/n$, we have that, except with probability at most $(e/4)^{24l^2/n}$, at most $48l^2/n$ corners are coloured blue during phases I, II and III.

Next we consider the number of blue corners drawn or removed during phases I, II and III. Assuming that there are never more than $48l^2/n$ blue corners in the urn during these $6l$ iterations, we have that the probability that a blue corner is drawn or removed in any iteration is at most $(48l^2/n)/(3n/2) = 32l^2/n^2$. Using Lemma 9.3 with $h = 6l$

and $p = 32l^2/n^2$, we have that, except with probability at most $(e/4)^{192l^3/n^2}$, at most $384l^3/n^2$ blue corners are drawn or removed during phases I, II and III. Thus, except with probability at most

$$(2/e)^{l/8} + (e/4)^{32l^2/3n} + (2/e)^{l^2/192n} + (e/4)^{24l^2/n} + (e/4)^{192l^3/n^2},$$

there are at least $l^2/192n - 384l^3/n^2 \geq l^2/384n$ blue corners in the urn throughout phase III.

Finally we consider the probability that no blue corner is drawn during phase III. Assuming that there are at least $l^2/384n$ blue corners in the urn throughout phase III, we have that the probability of drawing a blue corner at each iteration is at least $(l^2/384n)/3n = l^2/1152n^2$, and thus that the probability of not drawing a blue corner in any of these $2l$ iterations is at most $(1 - l^2/1152n^2)^{2l} \leq (1/e)^{l^3/576n^2}$. Thus, except with probability at most

$$(2/e)^{l/8} + (e/4)^{32l^2/3n} + (2/e)^{l^2/192n} + (e/4)^{24l^2/n} + (e/4)^{384l^3/n^2} + (1/e)^{l^3/576n^2},$$

either the cycle closes or a triply visited triangle is created before the length of the cycle reaches $12l$. These six contributions are all

$$\exp -\Omega\big((\log n)^3\big) = O\left(\frac{1}{n^2}\right).$$

□

Theorem 9.5: The expected number of cycles of order at most one is

$$\mathrm{Ex}[s_1] = \frac{2}{3}\log n + \log 3 + \frac{2\gamma}{3} + O\left(\frac{(\log n)^7}{n^{1/8}}\right).$$

Proof: Using Propositions 9.1, 9.22 and 9.4, it will suffice to show that

$$\sum_{m\leq k\leq 12l} \frac{1}{k} \sum_{0\leq j\leq k/4} \left[\binom{k-2j}{2j} + \binom{k-2j-1}{2j-1}\right] (2j-1)!! \times \\ \frac{3^{k-2j}\, n!}{(n-k+2j)!} \cdot \frac{(3n-k+j-1)!!}{(3n-1)!!} = \frac{1}{3}\log\frac{27n^2}{m^3} - \frac{\gamma}{3} + O\left(\frac{(\log n)^7}{n^{1/8}}\right), \qquad (2)$$

where $m = n^{5/8}$ and $l = \lceil n^{2/3}\log n\rceil$. To do this, we shall use a “bootstrapping” technique, whereby crude estimates are used to reduce the range of the summation over j, so that more delicate estimates will be applicable over the reduced range. Specifically, we shall show that for certain terms the inner summand in (9.2) is $O(1/n^3)$. Since there are $O(n^2)$

such terms, this will imply that the total contribution of these terms is $O(1/n)$, and thus can be neglected.

First, we shall show that all term with $j \geq 6a$, where

$$a = \frac{k^2}{6n},$$

may be neglected. Using the estimates

$$\binom{k-2j}{2j} + \binom{k-2j-1}{2j-1} \leq 2\binom{k-2j}{2j} \leq \frac{2\,k^{2j}}{(2j)!}, \tag{9.3}$$

$$(2j-1)!! = \frac{(2j)!}{2^j\, j!} \leq (2j)! \left(\frac{e}{2j}\right)^j, \tag{9.4}$$

$$\frac{3^{k-2j}\, n!}{(n-k+2j)!} \leq (3n)^{k-2j},$$

and

$$\begin{aligned} \frac{(3n-k+j-1)!!}{(3n-1)!!} &\leq \frac{1}{(3n)^{k-j}} \exp\left(\frac{k^2}{3n} + O\left(\frac{k}{n}\right) + O\left(\frac{k^3}{n^2}\right)\right) \\ &\leq \frac{1}{(3n)^{k-j}} \exp\left(2a + O\left((\log n)^3\right)\right), \end{aligned} \tag{9.5}$$

we have that the inner summand in (9.2) is at most

$$\left(\frac{ea}{j}\right)^j \exp\left(2a + O\left((\log n)^3\right)\right).$$

This is a decreasing function of j for $j \geq a$, and for $j = 6a$ it is equal to

$$\left(\frac{e^{4/3}}{6}\right)^{6a} \exp O\left((\log n)^3\right) = \exp\left(-\Omega(n^{1/4}) + O\left((\log n)^3\right)\right),$$

since $k \geq m = \Omega(n^{5/8})$. Thus all terms with $j \geq 6a$ are $O(1/n^3)$ and may therefore be neglected. Thus it will suffice to show that

$$\begin{aligned} &\sum_{m \leq k \leq 12l} \frac{1}{k} \sum_{0 \leq j \leq 6a} \left[\binom{k-2j}{2j} + \binom{k-2j-1}{2j-1}\right] (2j-1)!! \times \\ &\qquad \frac{3^{k-2j}\, n!}{(n-k+2j)!} \cdot \frac{(3n-k+j-1)!!}{(3n-1)!!} = \frac{1}{3} \log \frac{27n^2}{m^3} - \frac{\gamma}{3} + O\left(\frac{(\log n)^7}{n^{1/8}}\right). \end{aligned} \tag{9.6}$$

Next, we shall show that all terms with $j \le a - d$ or $j \ge a + d$, where

$$d = a^{1/2}(\log n)^2,$$

may be neglected. Using the estimates (9.3), (9.4),

$$\begin{aligned}
\frac{3^{k-2j}\, n!}{(n-k+2j)!} &\le (3n)^{k-2j} \exp\left(-\frac{(k-2j)^2}{2n} + O\left(\frac{k}{n}\right) + O\left(\frac{k^3}{n^2}\right)\right) \\
&\le (3n)^{k-2j} \exp\left(-\frac{k^2}{2n} + \frac{2kj}{n} + O\left(\frac{k}{n}\right) + O\left(\frac{k^3}{n^2}\right)\right) \\
&\le (3n)^{k-2j} \exp\left(-3a + O\left((\log n)^3\right)\right)
\end{aligned}$$

(where we have used $j \le 6a$) and (9.5), we have that the inner summand in (9.6) is at most

$$\begin{aligned}
\left(\frac{ea}{j}\right)^j \exp\left(-a + O\left((\log n)^3\right)\right) &= \exp\left(a\left(\frac{j}{a} - \frac{j}{a}\log\frac{j}{a} - 1\right) + O\left((\log n)^3\right)\right) \\
&\le \exp\left(-\frac{(j-a)^2}{6a} + O\left((\log n)^3\right)\right),
\end{aligned}$$

since $x - x\log x - 1 \le -(x-1)^2/6$ for $0 \le x \le 6$ and $j \le 6a$. Thus if $j \le a-d$ or $j \ge a+d$, we have that the inner summand in (9.6) is at most

$$\exp\left(-\frac{d^2}{6a} + O\left((\log n)^3\right)\right) = \exp\left(-\Omega\left((\log n)^4\right) + O\left((\log n)^3\right)\right).$$

Thus all terms with $j \le a - d$ or $j \ge a + d$ are $O(1/n^3)$ and may therefore be neglected. Thus it will suffice to show that

$$\begin{aligned}
&\sum_{m \le k \le 12l} \frac{1}{k} \sum_{a-d \le j \le a+d} \left[\binom{k-2j}{2j} + \binom{k-2j-1}{2j-1}\right] (2j-1)!! \times \\
&\qquad \frac{3^{k-2j}\, n!}{(n-k+2j)!} \cdot \frac{(3n-k+j-1)!!}{(3n-1)!!} = \frac{1}{3}\log\frac{27n^2}{m^3} - \frac{\gamma}{3} + O\left(\frac{(\log n)^7}{n^{1/8}}\right). \quad (9.7)
\end{aligned}$$

Finally, we have the estimates

$$\begin{aligned}
\binom{k-2j}{2j} + \binom{k-2j-1}{2j-1} &= \binom{k-2j}{2j} \exp\, O\left(\frac{j}{k}\right) \\
&= \frac{k^{2j}}{(2j)!} \exp\left(-\frac{6j^2}{k} + O\left(\frac{j}{k}\right) + O\left(\frac{j^3}{k^2}\right)\right) \\
&= \frac{k^{2j}}{(2j)!} \exp\left(-\frac{k^3}{6n^2} + O\left(\frac{(\log n)^4}{n^{1/6}}\right)\right)
\end{aligned}$$

(where we have used $j = a + O(d)$ and $k = O(l)$),

$$
\begin{aligned}
(2j-1)!! &= \frac{(2j)!}{2^j\, j!} \\
&= (2j)!\, \left(\frac{1}{2\pi j}\right)^{1/2} \left(\frac{e}{2j}\right)^j \exp\, O\left(\frac{1}{j}\right) \\
&= (2j)!\, \left(\frac{1}{2\pi a}\right)^{1/2} \left(\frac{e}{2j}\right)^j \exp\, O\left(\frac{d}{a}\right) \\
&= (2j)!\, \left(\frac{1}{2\pi a}\right)^{1/2} \left(\frac{e}{2j}\right)^j \exp\, O\left(\frac{(\log n)^2}{n^{1/8}}\right)
\end{aligned}
$$

(where we have used $j = a + O(d)$ and $k = \Omega(m)$),

$$
\begin{aligned}
&\frac{3^{k-2j}\, n!}{(n-k+2j)!} \\
&\quad = (3n)^{k-2j} \exp\left(-\frac{(k-2j)^2}{2n} + O\left(\frac{k}{n}\right) - \frac{(k-2j)^3}{6n^2} + O\left(\frac{k^2}{n^2}\right) + O\left(\frac{k^4}{n^3}\right)\right) \\
&\quad = (3n)^{k-2j} \exp\left(-\frac{k^2}{2n} + \frac{2kj}{n} - \frac{k^3}{6n^2} + O\left(\frac{k^4}{n^3}\right)\right) \\
&\quad = (3n)^{k-2j} \exp\left(-\frac{k^2}{2n} + \frac{k^3}{6n^2} + O\left(\frac{(\log n)^4}{n^{1/6}}\right)\right)
\end{aligned}
$$

(where we have used $j = a + O(d)$ and $k = O(l)$) and

$$
\begin{aligned}
&\frac{(3n-k+j-1)!!}{(3n-1)!!} \\
&\quad = \frac{1}{(3n)^{k-j}} \exp\left(\frac{(k-j)^2}{3n} + O\left(\frac{k}{n}\right) + \frac{2(k-j)^3}{27n^2} + O\left(\frac{k^2}{n^2}\right) + O\left(\frac{k^4}{n^3}\right)\right) \\
&\quad = \frac{1}{(3n)^{k-j}} \exp\left(\frac{k^2}{3n} - \frac{2kj}{3n} + \frac{2k^3}{27n^2} + O\left(\frac{k^4}{n^3}\right)\right) \\
&\quad = \frac{1}{(3n)^{k-j}} \exp\left(\frac{k^2}{3n} - \frac{k^3}{27n^2} + O\left(\frac{(\log n)^4}{n^{1/6}}\right)\right)
\end{aligned}
$$

(where we have used $j = a + O(d)$ and $k = O(l)$). Thus the inner summand in (9.7) is

$$
\begin{aligned}
&\left(\frac{1}{2\pi a}\right)^{1/2} \left(\frac{ea}{j}\right)^j \exp\left(-a - \frac{k^3}{27n^2} + O\left(\frac{(\log n)^2}{n^{1/8}}\right)\right) \\
&\qquad = \left(\frac{1}{2\pi a}\right)^{1/2} \exp\left(-\frac{(j-a)^2}{2a} + O\left(\frac{|j-a|^3}{a^2}\right) - \frac{k^3}{27n^2} + O\left(\frac{(\log n)^2}{n^{1/8}}\right)\right) \\
&\qquad = \left(\frac{1}{2\pi a}\right)^{1/2} \exp\left(-\frac{(j-a)^2}{2a} - \frac{k^3}{27n^2} + O\left(\frac{(\log n)^6}{n^{1/8}}\right)\right),
\end{aligned}
$$

since $x - x\log x - 1 = (x-1)^2/2 + O\big(|x-1|^3\big)$, $j = a + O(d)$ and $k \ge m = \Omega(n^{5/8})$. Thus the inner sum in (9.7) is

$$\left(\frac{1}{2\pi a}\right)^{1/2} \exp\left(-\frac{k^3}{27n^2} + O\left(\frac{(\log n)^6}{n^{1/8}}\right)\right) \sum_{a-d\le j\le a+d} \exp -\frac{(j-a)^2}{2a}.$$

We have

$$\begin{aligned}\sum_{a-d\le j\le a+d} \exp -\frac{(j-a)^2}{2a} &= \sum_{-d\le i\le d} \exp -\frac{i^2}{2a} \\ &= \int_{-d}^{d} \exp -\frac{\xi^2}{2a}\, d\xi + O(1),\end{aligned}$$

since the error in estimating a sum by the corresponding integral is at most the total variation of the summand. Furthermore, we have

$$\begin{aligned}\int_{-d}^{d} \exp -\frac{\xi^2}{2a}\, d\xi = a^{1/2} &\int_{-d/a^{1/2}}^{d/a^{1/2}} \exp -\frac{\eta^2}{2}\, d\eta \\ &= a^{1/2} \int_{-\infty}^{\infty} \exp -\frac{\eta^2}{2}\, d\eta + \exp -\Omega\big((\log n)^4\big) \\ &= (2\pi a)^{1/2} + \exp -\Omega\big((\log n)^4\big),\end{aligned}$$

since $\int_{-\infty}^{-\beta} \exp -\frac{\eta^2}{2}\, d\eta = \int_{\beta}^{\infty} \exp -\frac{\eta^2}{2}\, d\eta \le \frac{1}{\beta} \exp -\frac{\beta^2}{2}$, $d/a^{1/2} = (\log n)^2$ and $\int_{-\infty}^{\infty} \exp -\frac{\eta^2}{2}\, d\eta = (2\pi)^{1/2}$. Thus the inner sum in (9.7) is

$$\exp\left(-\frac{k^3}{27n^2} + O\left(\frac{(\log n)^6}{n^{1/8}}\right)\right).$$

Summing this result over $m \le k \le 12l$ yields

$$\exp\, O\left(\frac{(\log n)^6}{n^{1/8}}\right) \sum_{m\le k\le 12l} \frac{1}{k} \exp -\frac{k^3}{27n^2}. \tag{9.8}$$

Evaluating this sum in the same way as the one in Section 8, we have

$$\begin{aligned}\sum_{m\le k\le 12l} \frac{1}{k} \exp -\frac{k^3}{27n^2} &= \int_{m}^{12l} \frac{1}{\xi} \exp -\frac{\xi^3}{27n^2}\, d\xi + O\left(\frac{1}{n^{5/8}}\right) \\ &= \frac{1}{3} \int_{m^3/27n^2}^{(12l)^3/27n^2} \frac{1}{\eta} \exp -\eta\, d\eta + O\left(\frac{1}{n^{5/8}}\right) \\ &= \frac{1}{3} \log \frac{27n^2}{m^3} - \frac{\gamma}{3} + O\left(\frac{1}{n^{1/8}}\right),\end{aligned}$$

since $\int_\alpha^\infty \frac{1}{\eta} \exp -\eta \, d\eta = \log \frac{1}{\alpha} - \gamma + O(\alpha)$ for $\alpha \to 0$ and $\int_\beta^\infty \frac{1}{\eta} \exp -\eta \, d\eta \le \frac{1}{\beta} \exp -\beta$ for $\beta \to \infty$. Substituting these results into (9.8) yields (9.7) □

10. Conclusion

The principal problem left open by the present work is to determine completely the probability distribution for the random variable h. If, for example, this were done by giving the generating function for h, one could presumably obtain the moments by differentiation (as was done for h' and h'' in Section 2), and thus confirm or refute the conjectured asymptotics of $\mathrm{Ex}[h]$ and $\mathrm{Var}[h]$. The most promising approaches to this problem appear to lie in the connection with matrix models. An analogous model with a quartic interaction has been described by Bessis, Itzykson and Zuber [12], who relate the coefficients in a conjectured asymptotic expansion to the enumeration of certain graphs according to their genera. The existence of this asymptotic expansion and the interpretation of the coefficients has been rigourously established by Ercolini and McLaughlin [50]. Their approach does not, however, appear to provide a derivation of the results of this paper, let alone our more precise conjectures. Another approach would be to extend Harer and Zagier's [43] analysis of glueings of the $(2n)$-gon to n triangles (see also Penner [44] and Itzykson and Zuber [45]). This approach leads naturally to the ξ-fold integral

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\sum_{1 \le i \le \xi} x_i^3 \right)^n e^{-\frac{1}{2} \sum_{1 \le i \le \xi} x_i^2} \prod_{1 \le i < j \le \xi} (x_i - x_j)^2 \, dx_1 \cdots dx_\xi .$$

If this integral could be evaluated as an analytic function of ξ, the result would be (apart from an easily calculable normalizing factor) the desired generating function for h.

Another open problem is to determine the asymptotic behaviour of the expected number of complex boundary cycles in the fat-graph model. The heuristic analysis of Section 7 suggests that this number tends to a constant, $\log 4$, but we cannot rigourously exclude either that it grows unboundedly or tends to zero. Yet another problem is to analyze the effect of dropping that requirement that all surfaces be orientable; this could be done by allowing each glueing to occur in either of two ways with equal probability.

Finally, our model assigns equal probabilities to all possible glueings of the triangles, with the result that the most likely outcome is a connected surface of high genus. It would be of interest to explore the consequences of departing from this assumption. One could, for example, associate a self-interaction energy with cycles that doubly traverse ribbons. Such an interaction that penalized double traversals of ribbons would presumably tend to

increase the total Euler characteristic, increasing the number of components and decreasing their genera, with results that would be more similar to matrix models in the large-N limit.

11. Acknowlegdment

The rational function $R(s,q)$ certifying the identity in Section 6 was found using Marko Petkovsek's program `Zeil` in the Mathematica package `gosper.m` available at `http://www.cis.upenn.edu∼wilf/AeqB.html`.

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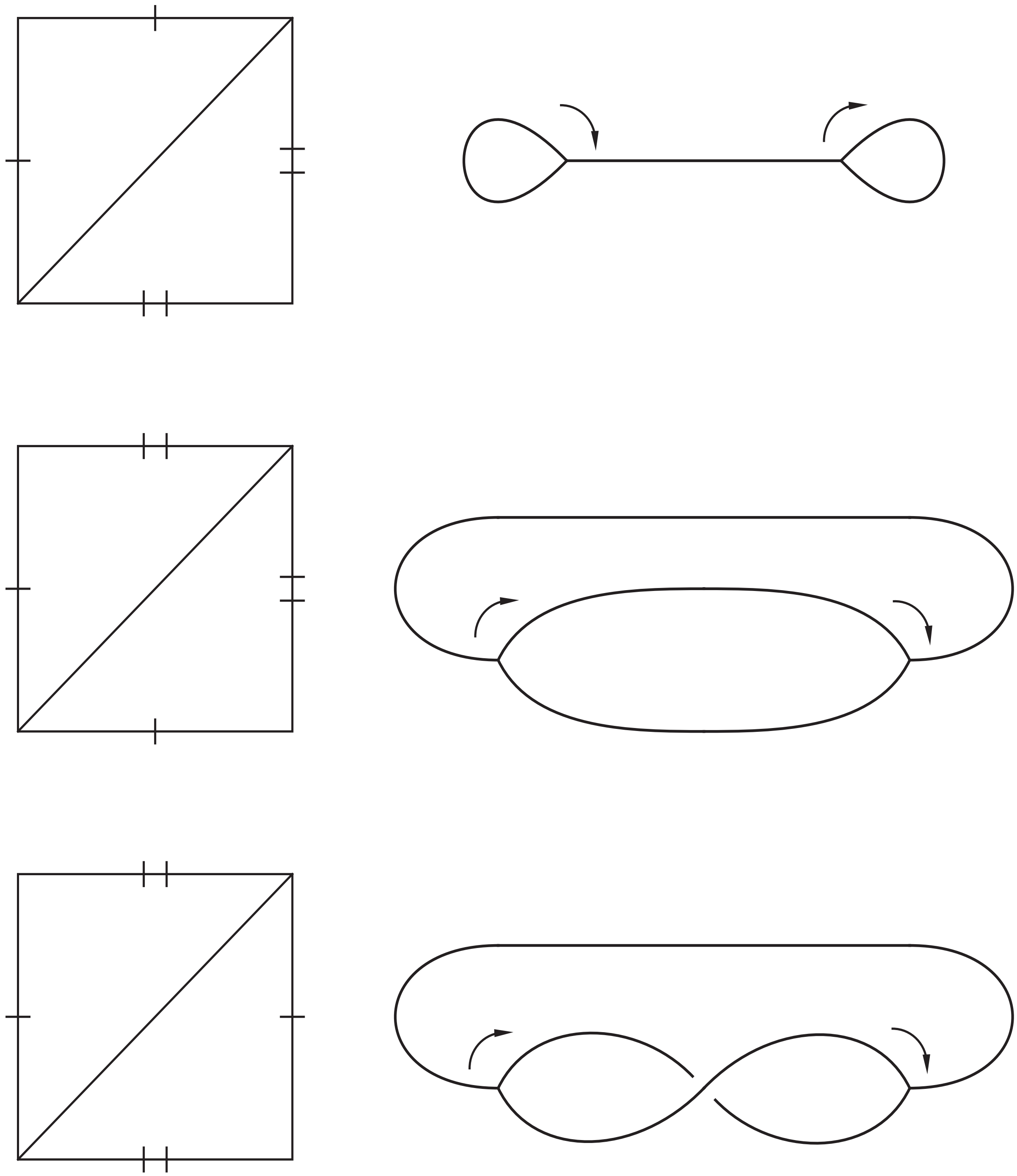

Figure 1

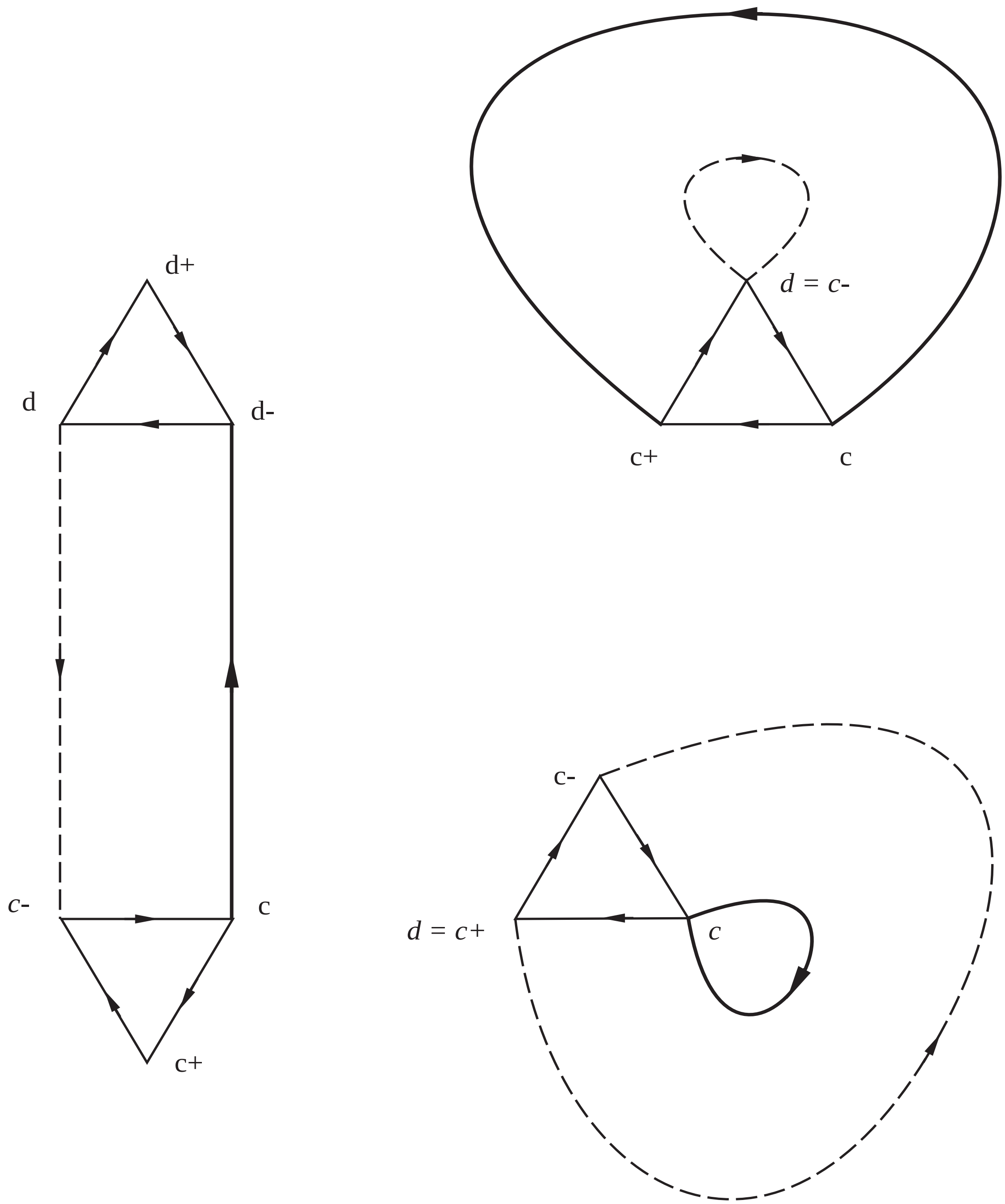

Figure 2

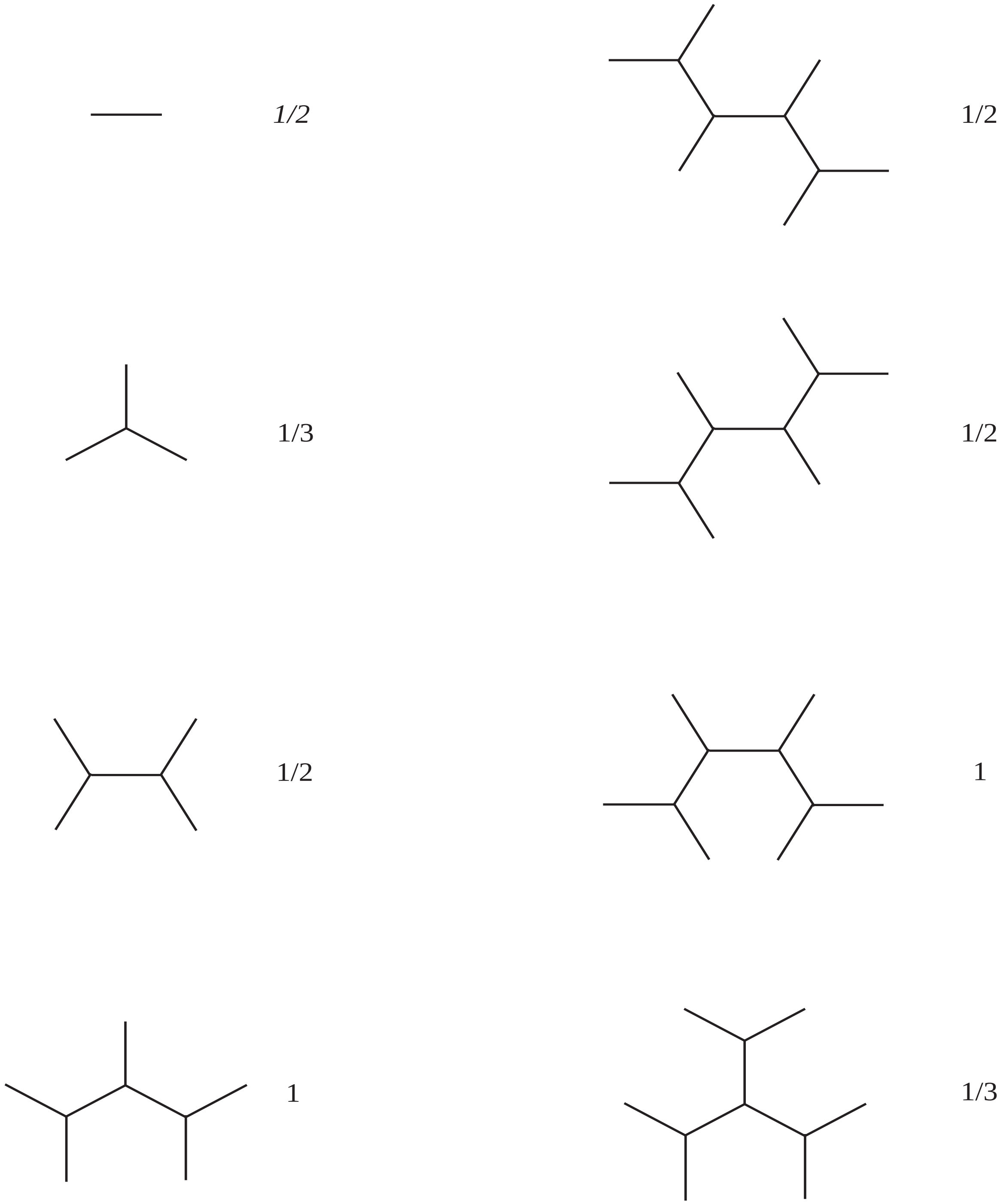

Figure 3